

SOBOLEV SPACES WITH RESPECT TO WEIGHTED GAUSSIAN MEASURES IN INFINITE DIMENSIONS

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ABSTRACT. Let X be a separable Banach space endowed with a non degenerate Gaussian measure μ and let w be a positive function on X such that $w \in W^{1,s}(X, \mu)$ and $\log w \in W^{1,t}(X, \mu)$ for some $s > 1$ and $t > s'$. We introduce and study weighted Sobolev functions and their traces on hypersurfaces of the form $\{x \in X \mid G(x) = 0\}$, where w is the chosen weight and G is a suitable version of Gaussian Sobolev function.

1. INTRODUCTION

Let X be a separable Banach space with norm $\|\cdot\|$, endowed with a nondegenerate centered Gaussian measure μ . The associated Cameron–Martin space is denoted by H , its inner product by $\langle \cdot, \cdot \rangle_H$ and its norm by $|\cdot|_H$. The covariance operator is denoted by $Q : X^* \rightarrow X$, where X^* is the topological dual of X . Let $\{e_k\}_{k \in \mathbb{N}}$ be an orthonormal basis of H , contained in $Q(X^*)$, and for every $k \in \mathbb{N}$ set $\widehat{e}_k = Q^{-1}(e_k)$. The spaces $W^{1,p}(X, \mu)$ and $W^{2,p}(X, \mu)$ are the classical Sobolev spaces of the Malliavin calculus (see [Bog98]).

The aim of this paper is to study weighted Sobolev functions and their traces on adequate hypersurfaces, namely the sets of the form $\{x \in X \mid G(x) = 0\}$ where $G : X \rightarrow \mathbb{R}$ is a suitable version of a Sobolev function. The main results about Gaussian Sobolev spaces can be found in [Bog98], while results about surface measures and traces of Sobolev functions in Gaussian Sobolev spaces can be found in [FdLP91, Fey01] and in [CL14], respectively.

In order to develop our theory we assume the following hypotheses about the weight:

Hypothesis 1.1. Throughout the paper we will denote the weight function by w and assume that $w(x) > 0$, μ -a.e. and

- (1) $w \in W^{1,s}(X, \mu)$ for some $s > 1$ and it is a $(1, s)$ -precise version (see section 2);
- (2) $\log w \in W^{1,t}(X, \mu)$ for some $t > s'$ (this technical hypothesis will be explained later).

Let $\nu = w\mu$ and recall that ν is a Radon measure absolutely continuous with respect to μ .

We will assume the following hypotheses about the “regularity” of the hypersurfaces we work with:

Hypothesis 1.2. Let $G \in W^{2,q}(X, \mu)$ be a $(2, q)$ -precise version (see Section 2) for every $q > 1$ and assume

- (1) $\mu(G^{-1}(-\infty, 0)) > 0$;
- (2) there exists $\delta > 0$ such that $|\nabla_H G|_H^{-1} \in L^q(G^{-1}(-\delta, \delta), \mu)$ for every $q > 1$.

Date: March 29, 2016.

2010 Mathematics Subject Classification. 28C20, 46G12.

Key words and phrases. Infinite dimensional analysis, Traces, Weighted Gaussian measure, divergence operator, Weighted Sobolev spaces, Sublevel sets.

Recall the following definitions (see [Bog98]): we say that a function $f : X \rightarrow \mathbb{R}$ is *differentiable along H at x* if there is $v \in H$ such that

$$\lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t} = \langle v, h \rangle_H \quad \text{uniformly for } h \in H \text{ with } |h|_H = 1.$$

In this case the vector $v \in H$ is unique and we set $\nabla_H f := v$, moreover for every $k \in \mathbb{N}$ the directional derivative exists

$$(1.1) \quad \partial_k f(x) := \lim_{t \rightarrow 0} \frac{f(x + te_k) - f(x)}{t} = \langle \nabla_H f(x), e_k \rangle_H.$$

Our main result is an infinite dimensional weighted version of the divergence theorem, namely:

Theorem 1.3. *Let $p \geq \frac{t}{t-s}$. For every $\varphi \in W^{1,p}(G^{-1}(-\infty, 0), \nu)$ and $k \in \mathbb{N}$ we have*

$$\int_{G^{-1}(-\infty, 0)} (\partial_k \varphi + \varphi \partial_k \log w - \varphi \widehat{e}_k) d\nu = \int_{G^{-1}(0)} \text{Tr}_{G^{-1}(0)}(\varphi \partial_k G) \frac{w}{|\nabla_H G|_H} d\rho.$$

Furthermore if $\Phi \in W^{1,p}(G^{-1}(-\infty, 0), \nu; H)$ then

$$\int_{G^{-1}(-\infty, 0)} \text{div}_\nu \Phi d\nu = \int_{G^{-1}(0)} \langle \text{Tr}_{G^{-1}(0)} \Phi, \text{Tr}_{G^{-1}(0)} \nabla_H G \rangle_H \frac{w}{|\nabla_H G|_H} d\rho,$$

where $\text{Tr}_{G^{-1}(0)} \Psi = \sum_{n=1}^{+\infty} (\text{Tr}_{G^{-1}(0)} \psi_n) e_n$ if $\Psi \in W^{1,p}(G^{-1}(-\infty, 0), \nu; H)$ and $\psi_n = \langle \Psi, e_n \rangle$.

The spaces $W^{1,p}(G^{-1}(-\infty, 0), \nu)$ and $W^{1,p}(G^{-1}(-\infty, 0), \nu; H)$ will be defined in section 6 and the trace operator Tr will be introduced in section 7.

The paper is organized in the following way: in Section 2 we will introduce the basic notations that we will use throughout the paper.

In Section 3 we will prove some results about density of important classes of functions in the space $L^p(X, \nu)$. Namely we will prove that the classes of the bounded and Lipschitzian functions and of the smooth cylindrical functions are dense in $L^p(X, \nu)$ for $p \geq 1$.

In Section 4 we will introduce the Sobolev weighted spaces $W^{1,p}(X, \nu)$ and study their basic properties. Namely we show that the following integration by parts formula holds:

$$(1.2) \quad \int_X \partial_h f(x) d\nu(x) = \int_X f(x) (\widehat{h}(x) - \partial_h \log w(x)) d\nu(x).$$

This equality allows us to prove the $\nabla_H : \mathcal{FC}_b^\infty(X) \rightarrow L^p(X, \nu)$ is a closable operator whether $p \geq \frac{t}{t-s}$, where $\mathcal{FC}_b^\infty(X)$ denotes the *smooth cylindrical functions* (the functions of the form $f(x) = \varphi(l_1(x), \dots, l_n(x))$, for some $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, $l_1, \dots, l_n \in X^*$ and $n \in \mathbb{N}$). Some of the results of this section can be found in [AM88], where a more general class of weights were considered on the space $\mathcal{C}[0, 1]$.

In Section 5 we will introduce the divergence operator div_ν as minus the formal adjoint of the gradient operator along H and investigate some of its basic properties. In Proposition 5.3 we will prove that under suitable hypotheses on the weight (namely Equation (5.3)) then $W^{1,2}(X, \nu; H)$ is contained in the domain of div_ν and $\text{div}_\nu \Phi \in L^2(X, \nu)$ for every $\Phi \in W^{1,2}(X, \nu; H)$. Example 5.4 shows that the conditions in Proposition 5.3 are only sufficient to ensure that the domain of div_ν is not empty. Finally Proposition 5.5 shows that if $\Phi \in W^{1,q}(X, \mu; H)$ (the unweighted Sobolev spaces) then $\text{div}_\nu \Phi$ belongs to $L^p(X, \nu)$ for

some reasonable values of p . Furthermore an explicit formula for the calculation of div_ν is given in Equation (5.7).

In Section 6 we will introduce Sobolev spaces on sets of the form $G^{-1}(-\infty, 0)$ in a similar way as in Section 4, where G is a function satisfying Hypothesis 1.2.

In Section 7 we will introduce the trace $\operatorname{Tr}_{G^{-1}(0)} f$ of a Sobolev function f in $W^{1,p}(G^{-1}(-\infty, 0), \nu)$ on $G^{-1}(0)$, where G is a function satisfying Hypothesis 1.2. In this section we give the proof of our main result (Theorem 1.3) and we show in Proposition 7.5, that if $\bar{\varphi}$ is a $(1, p)$ -precise version (see Section 2) of an element φ in $W^{1,p}(G^{-1}(-\infty, 0), \nu)$, then

$$\operatorname{Tr}_{G^{-1}(0)} \varphi = \bar{\varphi}|_{G^{-1}(0)} \quad \rho\text{-a.e.}$$

where ρ is the Feyel–de La Pradelle Hausdorff–Gauss surface measure introduced in [Fey01] (see also Section 6 for more informations).

In Section 8 we will show how our results can be applied to explicit examples. The chosen hypersurface will be the unit sphere and the hyperplanes. In Example 8.1 we study the weights $w_\lambda(x) = \exp(\lambda\|x\|_X^2)$, where X is a separable Hilbert space and λ is a real number. In example 8.2 we study the weights $w_q(x) = \exp(\|x\|_q)$ in ℓ_2 , where $q > 1$ and

$$\|x\|_q = \left(\sum_{i=1}^{+\infty} |x(i)|^q \right)^{\frac{1}{q}}.$$

We show that every w_q satisfies Hypothesis 1.1 (note that $\|\cdot\|_q$ is not continuous if $1 < q < 2$). Eventually in Example 8.3 we study the weight $w(f) = \exp(\|f\|_\infty)$ in the space $\mathcal{C}[0, 1]$. In this case we will be interested in two type of surfaces: the hyperplanes and $\{f \in \mathcal{C}[0, 1] \mid \|f\|_2 = 1\}$. All the examples are concluded by some observations about the continuity of the trace operator from $W^{1,p}(G^{-1}(-\infty, 0), \nu)$ to $L^q(G^{-1}(0), w\rho)$ for $q \in [1, p]$, where ρ is the Feyel–de La Pradelle Hausdorff–Gauss surface measure.

2. NOTATIONS AND PRELIMINARIES

Recall that the Gaussian Sobolev spaces $W^{1,p}(X, \mu)$ and $W^{2,p}(X, \mu)$ with $p \geq 1$ (see [Bog98, Section 5.2]) are the completions of the smooth cylindrical functions $\mathcal{FC}_b^\infty(X)$ in the norms

$$\begin{aligned} \|f\|_{W^{1,p}(X, \mu)} &:= \|f\|_{L^p(X, \mu)} + \left(\int_X |\nabla_H f(x)|_H^p d\mu(x) \right)^{\frac{1}{p}}; \\ \|f\|_{W^{2,p}(X, \mu)} &:= \|f\|_{W^{1,p}(X, \mu)} + \left(\int_X \left(\sum_{k=1}^{+\infty} \sum_{h=1}^{+\infty} (\partial_{hk} f(x))^2 \right)^{\frac{p}{2}} d\mu(x) \right)^{\frac{1}{p}}. \end{aligned}$$

Such spaces can be identified with subspaces of $L^p(X, \mu)$. The (generalized) derivatives along H , $\nabla_H f$ and $\nabla_H^2 f$, are well defined and belong to $L^p(X, \mu; H)$ and $L^p(X, \mu; \mathcal{H}_2)$ respectively, where \mathcal{H}_2 is the set of the Hilbert–Schmidt bilinear forms in H (see [DU77]). The spaces $W^{1,p}(X, \mu; H)$ are defined in a similar way, replacing smooth cylindrical functions with H -valued smooth cylindrical functions (i.e. the linear span of the functions $x \mapsto f(x)h$, where f is a smooth cylindrical function and $h \in H$).

We shall use the Gaussian integration by parts formula (see [Bog98]) for $\varphi \in W^{1,p}(X, \mu)$ with $p > 1$:

$$(2.1) \quad \int_X \partial_k \varphi d\mu = \int_X \varphi \widehat{e}_k d\mu \quad \text{for every } k \in \mathbb{N}.$$

For every measurable map $\Phi : X \rightarrow X$ and for every $f \in \mathcal{FC}_b^\infty(X)$ we define

$$(2.2) \quad \partial_\Phi f(x) = \lim_{t \rightarrow 0} \frac{f(x + t\Phi(x)) - f(x)}{t},$$

whenever such a limit exists. If the limit exists μ -a.e in X and a function $\beta \in L^1(X, \mu)$ satisfies

$$\int_X \partial_\Phi f(x) d\mu(x) = - \int_X f(x) \beta(x) d\mu(x)$$

for every $f \in \mathcal{FC}_b^\infty(X)$, then β is called *Gaussian divergence of Φ* . Furthermore if β exists, then it is unique and it will be denoted by $\text{div}_\mu \Phi := \beta$. The Gaussian divergence is a linear bounded operator from $W^{1,p}(X, \mu; H)$ to $L^p(X, \mu)$, for every $p > 1$. Furthermore if $\Phi(x) = \sum_{k=1}^{+\infty} \varphi_k(x) e_k$, then

$$\text{div}_\mu \Phi = \sum_{k=1}^{+\infty} (\partial_k \varphi_k - \varphi_k \widehat{e}_k),$$

and if $\Phi \in W^{1,p}(X, \mu; H)$ and $f \in W^{1,p'}(X, \mu)$ we have

$$\int_X \langle \nabla_H f, \Phi \rangle d\mu = - \int_X f \text{div}_\mu \Phi d\mu.$$

For further informations see [Bog98, section 5.8].

Let L_p the infinitesimal generator of the *Ornstein–Uhlenbeck semigroup* in $L^p(X, \mu)$

$$T(t)f(x) := \int_X f\left(e^{-t}x + (1 - e^{-2t})^{\frac{1}{2}}y\right) d\mu(y) \quad \text{for } t > 0.$$

For $k = 1, 2$, the $C_{k,p}$ -capacity of an open set $A \subseteq X$ is

$$C_{k,p}(A) := \inf \left\{ \|f\|_{L^p(X, \mu)} \mid (I - L_p)^{-\frac{k}{2}} f \geq 1 \text{ } \mu\text{-a.e. in } A \right\}.$$

For a general Borel set $B \subseteq X$ we let $C_{k,p}(B) = \inf \{C_{k,p}(A) \mid B \subseteq A \text{ open}\}$. Let $f \in W^{k,p}(X, \mu)$, f is an equivalence class of functions and we call every element “version”. A version \bar{f} of f exists that is Borel measurable and $C_{k,p}$ -quasicontinuous, i.e. for every $\varepsilon > 0$ there exists an open set $A \subseteq X$ such that $C_{k,p}(A) \leq \varepsilon$ and $\bar{f}|_{X \setminus A}$ is continuous. Furthermore for every $r > 0$

$$C_{k,p}(\{x \in X \mid |\bar{f}(x)| > r\}) \leq \frac{1}{r} \left\| (I - L_p)^{-\frac{k}{2}} \bar{f} \right\|_{L^p(X, \mu)}.$$

See [Bog98, Theorem 5.9.6]. Such a version is called a (k, p) -precise version of f . Two precise versions of the same f coincide outside sets with null $C_{k,p}$ -capacity. All our results will be independent on our choice of a precise version of f .

3. DENSITY OF SOME CLASSES OF FUNCTIONS

In the finite dimensional case it is possible to define the Sobolev spaces in many different equivalent ways. In infinite dimension it is still not clear whether different reasonable definitions are equivalent (except the case of Gaussian measures on the whole space, see [Bog98], on convex subsets, see [Hin03], and on a generalization of convex subsets, see [Hin11]). Our approach is to define the Sobolev space as the domain of the closure of a “gradient” operator, and in order to do so we have to define such an operator on a dense class of “good” functions. We devote this section to show that some meaningful classes of functions are dense in $L^p(X, \nu)$ for every $p \geq 1$.

The results of this section should be more or less known, but we were unable to find suitable references in the literature.

Lemma 3.1. *Let X be a topological normal space, let γ be a finite Radon measure on X and let $f : X \rightarrow \mathbb{R}$ be a Borel function. For every $\varepsilon > 0$ there exists $g_\varepsilon : X \rightarrow \mathbb{R}$ bounded and continuous such that*

$$\gamma(\{x \in X \mid f(x) \neq g_\varepsilon(x)\}) < \varepsilon.$$

Furthermore $\sup_{x \in X} |g_\varepsilon(x)| \leq \sup_{x \in X} |f(x)|$.

Proof. Consider a compact set $K_0 \subseteq X$ such that $\gamma(X \setminus K_0) < \frac{\varepsilon}{2}$. The function $f|_{K_0} : K_0 \rightarrow \mathbb{R}$ is a Borel function and by Lusin’s theorem (see [Rud87, Theorem 2.24]) there exists a continuous function $\tilde{g}_\varepsilon : K_0 \rightarrow \mathbb{R}$ such that

$$\gamma\left(\left\{x \in K_0 \mid f|_{K_0}(x) \neq \tilde{g}_\varepsilon(x)\right\}\right) < \frac{\varepsilon}{2}$$

and

$$\sup_{x \in K_0} |\tilde{g}_\varepsilon(x)| \leq \sup_{x \in K_0} |f|_{K_0}(x)| = \sup_{x \in K_0} |f(x)|.$$

By Tietze’s theorem (see [Dug78, Theorem 5.1]) there exists a function $g_\varepsilon : X \rightarrow \mathbb{R}$ such that

- (1) $g_\varepsilon(x) = \tilde{g}_\varepsilon(x)$, for every $x \in K_0$;
- (2) $\sup_{x \in X} |g_\varepsilon(x)| \leq \sup_{x \in K_0} |\tilde{g}_\varepsilon(x)|$;
- (3) g_ε is a continuous function.

We obtain

$$\sup_{x \in X} |g_\varepsilon(x)| \leq \sup_{x \in K_0} |\tilde{g}_\varepsilon(x)| \leq \sup_{x \in K_0} |f(x)| \leq \sup_{x \in X} |f(x)|$$

and

$$\begin{aligned} \gamma(\{x \in X \mid f(x) \neq g_\varepsilon(x)\}) &\leq \gamma(X \setminus K_0) + \gamma(\{x \in K_0 \mid f(x) \neq g_\varepsilon(x)\}) = \\ &= \gamma(X \setminus K_0) + \gamma\left(\left\{x \in K_0 \mid f|_{K_0}(x) \neq \tilde{g}_\varepsilon(x)\right\}\right) < \varepsilon. \end{aligned}$$

□

Observe that the conclusion of Lemma 3.1 holds even if the function f is unbounded. Obviously, in that case, the furthermore part is useless.

Proposition 3.2. *Let X be a topological normal space, let γ be a finite Radon measure on X , and $p \geq 1$. $\mathcal{C}_b(X)$ is dense in $L^p(X, \gamma)$.*

Proof. Let f be a fixed version of an element of $L^p(X, \gamma)$ and consider

$$\tilde{f}_k(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq k; \\ 0 & \text{if } |f(x)| > k. \end{cases}$$

Fix $\varepsilon > 0$ and let $f_k \in \mathcal{C}_b(X)$ such that

$$\gamma\left(\left\{x \in X \mid \tilde{f}_k(x) \neq f_k(x)\right\}\right) < \frac{\varepsilon^p}{2^{p+1}k^p} \quad \text{and} \quad \sup_{x \in X} |f_k(x)| \leq k,$$

obtained applying Lemma 3.1. By the Lebesgue dominated convergence theorem we get that there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\int_{\{|f|>k\}} |f|^p d\gamma < \left(\frac{\varepsilon}{2}\right)^p \quad \text{for every } k \geq k_\varepsilon.$$

If $k \geq k_\varepsilon$ then

$$\begin{aligned} \|f_k - f\|_{L^p(X, \gamma)} &\leq \|f_k - \tilde{f}_k\|_{L^p(X, \gamma)} + \|\tilde{f}_k - f\|_{L^p(X, \gamma)} = \\ &= \left(\int_{\{f_k \neq \tilde{f}_k\}} |f_k(x) - \tilde{f}_k(x)|^p d\gamma(x)\right)^{\frac{1}{p}} + \left(\int_{\{|f|>k\}} |f(x)|^p d\gamma(x)\right)^{\frac{1}{p}} < \varepsilon. \end{aligned}$$

□

The following Lemma shows that in the metric case, the function g_ε of Lemma 3.1 may be taken uniformly continuous and bounded.

Lemma 3.3. *Let X be a metric space, γ be a finite Radon measure on X and $f : X \rightarrow \mathbb{R}$ be a Borel function. For every $\varepsilon > 0$ there exists a bounded uniformly continuous function $g_\varepsilon : X \rightarrow \mathbb{R}$ (i.e. $g_\varepsilon \in \text{BUC}(X)$) such that*

$$\gamma(\{x \in X \mid f(x) \neq g_\varepsilon(x)\}) < \varepsilon.$$

Furthermore $\sup_{x \in X} |g_\varepsilon(x)| \leq 2 \sup_{x \in X} |f(x)|$.

Proof. Consider a compact set $K_0 \subseteq X$ such that $\gamma(X \setminus K_0) < \frac{\varepsilon}{2}$. The function $f|_{K_0} : K_0 \rightarrow \mathbb{R}$ is a Borel function and by Lusin's theorem (see [Rud87, Theorem 2.24]) there exists a continuous function $\tilde{g}_\varepsilon : K_0 \rightarrow \mathbb{R}$ such that

$$\gamma\left(\left\{x \in K_0 \mid f|_{K_0}(x) \neq \tilde{g}_\varepsilon(x)\right\}\right) < \frac{\varepsilon}{2}$$

and

$$\sup_{x \in K_0} |\tilde{g}_\varepsilon(x)| \leq \sup_{x \in K_0} |f|_{K_0}(x) = \sup_{x \in K_0} |f(x)|.$$

The Heine–Cantor theorem says that $\tilde{g}_\varepsilon \in \text{BUC}(K_0)$. Consider the $\text{BUC}(X)$ extension (see [Man90])

$$g_\varepsilon(x) = \begin{cases} \tilde{g}_\varepsilon(x) & x \in K_0; \\ \inf_{y \in K_0} \tilde{g}_\varepsilon(y) \frac{d(x, y)}{\text{dist}(x, K_0)} & x \notin K_0. \end{cases}$$

An easy computation gives that for every $x \notin K_0$

$$|g_\varepsilon(x)| \leq \sup_{z \in X} |f(z)|.$$

Eventually we get

$$\sup_{x \in X} |g_\varepsilon(x)| \leq \sup_{x \in K_0} |g_\varepsilon(x)| + \sup_{x \in X \setminus K_0} |g_\varepsilon(x)| \leq \sup_{x \in K_0} |\tilde{g}_\varepsilon(x)| + \sup_{x \in X \setminus K_0} |g_\varepsilon(x)| \leq 2 \sup_{x \in X} |f(x)|.$$

Furthermore

$$\begin{aligned} \gamma(\{x \in X \mid f(x) \neq g_\varepsilon(x)\}) &\leq \gamma(X \setminus K_0) + \gamma(\{x \in K_0 \mid f(x) \neq g_\varepsilon(x)\}) = \\ &= \gamma(X \setminus K_0) + \gamma\left(\left\{x \in K_0 \mid f|_{K_0}(x) \neq \tilde{g}_\varepsilon(x)\right\}\right) < \varepsilon. \end{aligned}$$

□

Proposition 3.4. *Let X be a metric space, let γ be a finite Radon measure on X and $p \geq 1$. $\text{Lip}_b(X)$ is dense in $L^p(X, \gamma)$, where $\text{Lip}_b(X)$ is the set of bounded and Lipschitz functions on X .*

Proof. Let f be a version of a function in $L^p(X, \gamma)$. For every $k \in \mathbb{N}$ set

$$f_k(x) = \begin{cases} k & f(x) > k; \\ f(x) & -k \leq f(x) \leq k; \\ -k & f(x) < -k. \end{cases}$$

Applying Lemma 3.3, for every $k \in \mathbb{N}$ we can construct a function $\tilde{f}_k \in \text{BUC}(X)$ such that

$$\gamma\left(\left\{x \in X \mid \tilde{f}_k(x) \neq f_k(x)\right\}\right) \leq \frac{1}{2^k},$$

and $\sup_{x \in X} |\tilde{f}_k(x)| \leq 2 \sup_{x \in X} |f_k(x)| \leq 2k$. [Mic03, Theorem 1] gives a function $g_k \in \text{Lip}_b(X)$ such that

$$\|g_k - \tilde{f}_k\|_\infty \leq \frac{1}{2^k}.$$

We have

$$\|g_k - f\|_{L^p(X, \gamma)} \leq \|g_k - \tilde{f}_k\|_{L^p(X, \gamma)} + \|\tilde{f}_k - f_k\|_{L^p(X, \gamma)} + \|f_k - f\|_{L^p(X, \gamma)}.$$

Observe that f_k converges pointwise γ -a.e. to f and $|f_k| \leq |f|$, then by the Lebesgue dominated convergence theorem we get $\lim_{k \rightarrow +\infty} \|f_k - f\|_{L^p(X, \gamma)} = 0$. Furthermore

$$\|g_k - \tilde{f}_k\|_{L^p(X, \gamma)} \leq \|g_k - \tilde{f}_k\|_\infty (\gamma(X))^{\frac{1}{p}} \leq \frac{(\gamma(X))^{\frac{1}{p}}}{2^k} \xrightarrow{k \rightarrow +\infty} 0,$$

and

$$\begin{aligned} \|\tilde{f}_k - f_k\|_{L^p(X, \gamma)} &= \left(\int_{\{\tilde{f}_k \neq f_k\}} |\tilde{f}_k(x) - f_k(x)|^p d\gamma(x) \right)^{\frac{1}{p}} \leq \\ &\leq 2^{\frac{p-1}{p}} (2^p + 1)^{\frac{1}{p}} k \left(\gamma\left(\left\{x \in X \mid \tilde{f}_k(x) \neq f_k(x)\right\}\right) \right)^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}} (2^p + 1)^{\frac{1}{p}} \frac{k}{2^{\frac{k}{p}}} \xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

□

Corollary 3.5. *Let X be a separable Banach space, let μ be a Gaussian measure on X and let w be a weight satisfying Hypothesis 1.1. If Y is a subset of X , then $\text{Lip}(Y)$ is dense in $L^p(Y, \nu)$ for every $p \geq 1$.*

Proof. First of all, observe that $\text{Lip}(Y) \subseteq L^p(Y, \nu)$ for every $p \geq 1$. Indeed let $\varphi \in \text{Lip}(Y)$ and fix $x_0 \in Y$, then

$$\begin{aligned} \int_Y |\varphi(x)|^p d\nu(x) &\leq 2^{p-1} \left(\int_Y |\varphi(x) - \varphi(x_0)|^p + |\varphi(x_0)|^p d\nu(x) \right) \leq \\ &\leq 2^{p-1} \left(K^p \int_Y \|x - x_0\|^p d\nu(x) + \int_X |\varphi(x_0)|^p d\nu(x) \right) \leq \\ &\leq 2^{p-1} \left(2^{p-1} K^p \left(\int_X \|x\|^p d\nu(x) + \|x_0\|^p \nu(X) \right) + |\varphi(x_0)|^p \nu(X) \right) \leq \\ &\leq 2^{p-1} \left(2^{p-1} K^p \left(\left(\int_X \|x\|^{ps'} d\mu(x) \right)^{\frac{1}{s'}} \left(\int_X w^s d\mu \right)^{\frac{1}{s}} + \|x_0\|^p \nu(X) \right) + |\varphi(x_0)|^p \nu(X) \right), \end{aligned}$$

where K is the Lipschitz constant of φ . The last term is finite due to Fernique's theorem (see [Bog98, Theorem 2.8.5]). The statement follows by Proposition 3.4. \square

Proposition 3.6. *Let X be a separable Banach space, μ be a Gaussian measure on X and w a weight satisfying Hypothesis 1.1. $\mathcal{FC}_b^\infty(X)$ is dense in $L^p(X, \nu)$ for every $p \geq 1$.*

Proof. By the Lebesgue dominated convergence theorem any $f \in \mathcal{C}_b(X)$ may be approached in $L^p(X, \gamma)$ by the sequence $f_n(x) := f(P_n x)$, where

$$P_n(x) = \sum_{i=1}^n \widehat{e}_i(x) e_i.$$

Recall that $P_n(x)$ converges pointwise ν -a.e. to x (see [Bog98, Theorem 3.5.1]). In its turn, the cylindrical functions f_n , are approached by their (finite dimensional) convolutions with smooth mollifiers that belongs to $\mathcal{FC}_b^\infty(X)$. \square

4. WEIGHTED SOBOLEV SPACES

We want to define the Sobolev space $W^{1,p}(X, \nu)$ as the domain of the closure of the gradient operator along H . A natural procedure is to prove preliminary an integration by parts formula.

Lemma 4.1. *Let $f \in \mathcal{FC}_b^\infty(X)$ and $h \in H$. The following formula holds:*

$$\int_X \partial_h f(x) d\nu(x) = \int_X f(x) (\widehat{h}(x) - \partial_h \log w(x)) d\nu(x).$$

Proof. Using the integration by parts formula for the Gaussian measure (Equation (2.1)) we get

$$\begin{aligned} \int_X \partial_h f(x) d\nu(x) &= \int_X \partial_h f(x) w(x) d\mu(x) = \int_X \partial_h (f(x) w(x)) d\mu(x) - \int_X f(x) \partial_h w(x) d\mu(x) = \\ &= \int_X \widehat{h}(x) f(x) w(x) d\mu(x) - \int_X f(x) \frac{\partial_h w(x)}{w(x)} w(x) d\mu(x) = \int_X f(x) (\widehat{h}(x) - \partial_h \log w(x)) d\nu(x). \end{aligned}$$

\square

We are now ready to prove the closability of the gradient operator along H .

Proposition 4.2. *The operator $\nabla_H : \mathcal{FC}_b^\infty(X) \rightarrow L^p(X, \nu; H)$ is closable in $L^p(X, \nu)$, whenever $p \geq \frac{t}{t-s'}$.*

Proof. Let $(f_k)_{k \in \mathbb{N}} \subseteq \mathcal{FC}_b^\infty(X)$ be such that

$$\begin{aligned} \lim_{k \rightarrow +\infty} f_k &= 0 && \text{in } L^p(X, \nu); \\ \lim_{k \rightarrow +\infty} \nabla_H f_k &= \Phi && \text{in } L^p(X, \nu; H). \end{aligned}$$

We want to prove that $\Phi(x) = 0$, ν -a.e. To this aim we will show that

$$\int_X \langle \Phi(x), e_n \rangle_H u(x) d\nu(x) = 0$$

for every $n \in \mathbb{N}$ and $u \in \mathcal{FC}_b^\infty(X)$. Recall that if $f, g \in \mathcal{FC}_b^\infty(X)$, then $fg \in \mathcal{FC}_b^\infty(X)$. By the integration by parts formula (Lemma 4.1), we get

$$(4.1) \quad \int_X \partial_n f_k(x) u(x) d\nu(x) = \int_X f_k(x) (\widehat{e}_n(x) - \partial_n \log w(x)) u(x) d\nu(x) - \int_X f_k(x) \partial_n u(x) d\nu(x).$$

By the Hölder inequality we get

$$\begin{aligned} & \int_X |\partial_n f_k(x) - \langle \Phi(x), e_n \rangle| |u(x)| d\nu(x) \leq \\ & \leq \left(\int_X |\partial_n f_k(x) - \langle \Phi(x), e_n \rangle|^p d\nu(x) \right)^{\frac{1}{p}} \left(\int_X |u(x)|^{p'} d\nu(x) \right)^{\frac{1}{p'}} \xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

Furthermore

$$\int_X |f_k(x) \partial_n u(x)| d\nu(x) \leq \left(\int_X |f_k(x)|^p d\nu(x) \right)^{\frac{1}{p}} \left(\int_X |\partial_n u(x)|^{p'} d\nu(x) \right)^{\frac{1}{p'}} \xrightarrow{k \rightarrow +\infty} 0.$$

Lastly

$$\begin{aligned} & \int_X |f_k| |\widehat{e}_n - \partial_n \log w| |u| d\nu \leq \|u\|_\infty \int_X |f_k| |\widehat{e}_n - \partial_n \log w| d\nu \leq_{(1)} \\ & \leq_{(1)} \|u\|_\infty \left(\int_X |f_k|^p d\nu \right)^{\frac{1}{p}} \left(\int_X |\widehat{e}_n - \partial_n \log w|^{p'} w d\mu \right)^{\frac{1}{p'}} \leq_{(2)} \\ & \leq_{(2)} \|u\|_\infty \left(\int_X |f_k|^p d\nu \right)^{\frac{1}{p}} \left(\int_X w^s d\mu \right)^{\frac{1}{s}} \left(\int_X |\widehat{e}_n - \partial_n \log w|^{p's'} d\mu \right)^{\frac{1}{p's'}} \xrightarrow{k \rightarrow +\infty} 0; \end{aligned}$$

where both (1) and (2) follow applying the Hölder inequality. Note that the last integral is finite whenever $p's' \leq t$, i.e. $p \geq \frac{t}{t-s'}$ which is an assumption. Letting $k \rightarrow +\infty$ in Equation (4.1) we get

$$\int_X \langle \Phi(x), e_n \rangle u(x) d\nu(x) = 0$$

for every $n \in \mathbb{N}$ and $u \in \mathcal{FC}_b^\infty(X)$. □

Definition 4.3 (Weighted Sobolev spaces). Let $p \geq \frac{t}{t-s'}$. We denote by $W^{1,p}(X, \nu)$ the domain of the closure of ∇_H (which we still denote by the symbol ∇_H) in $L^p(X, \nu)$. It is a Banach space with the graph norm

$$\|f\|_{W^{1,p}(X, \nu)} = \left(\int_X |f(x)|^p d\nu(x) \right)^{\frac{1}{p}} + \left(\int_X |\nabla_H f(x)|_H^p d\nu(x) \right)^{\frac{1}{p}}.$$

In the same way we define $W^{1,p}(X, \nu; H)$ using H -valued smooth cylindrical functions. Furthermore Lemma 4.1 holds for every $f \in W^{1,p}(X, \nu)$.

The first thing we want to show is the following result about the coincidence of the closure of the gradient operator along H in $L^p(X, \nu)$ and $L^q(X, \mu)$.

Proposition 4.4. *In this proposition we will denote by $\nabla_H^{\mu,q}$ and $\nabla_H^{\nu,p}$ the closure of the gradient operator along H in $L^q(X, \mu)$ and $L^p(X, \nu)$, respectively. Let $p \geq \frac{t}{t-s'}$ and $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{FC}_b^\infty(X)$. If there exists $f \in W^{1,q}(X, \mu)$, for some $q > ps'$, such that*

$$\begin{aligned} \lim_{n \rightarrow +\infty} f_n &= f && \text{in } L^q(X, \mu); \\ \lim_{n \rightarrow +\infty} \nabla_H f_n &= \nabla_H^{\mu,q} f && \text{in } L^q(X, \mu; H), \end{aligned}$$

then

$$\begin{aligned} \lim_{n \rightarrow +\infty} f_n &= f && \text{in } L^p(X, \nu); \\ \lim_{n \rightarrow +\infty} \nabla_H f_n &= \nabla_H^{\nu,p} f && \text{in } L^p(X, \nu; H). \end{aligned}$$

Furthermore $\nabla_H^{\nu,p} f = \nabla_H^{\mu,q} f$ μ -a.e.

Proof. By the Hölder inequality we get

$$\begin{aligned} \int_X |f_n - f|^p d\nu &\leq \left(\int_X w^s d\mu \right)^{\frac{1}{s}} \left(\int_X |f_n - f|^{ps'} d\mu \right)^{\frac{1}{s'}} \leq \\ &\leq \left(\int_X w^s d\mu \right)^{\frac{1}{s}} \left(\int_X |f_n - f|^q d\mu \right)^{\frac{p}{q}} \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

So $\lim_{n \rightarrow +\infty} f_n = f$ in $L^p(X, \nu)$. In the same way

$$\begin{aligned} \int_X |\nabla_H f_n - \nabla_H^{\mu,q} f|_H^p d\nu &\leq \left(\int_X w^s d\mu \right)^{\frac{1}{s}} \left(\int_X |\nabla_H f_n - \nabla_H^{\mu,q} f|_H^{ps'} d\mu \right)^{\frac{1}{s'}} \leq \\ &\leq \left(\int_X w^s d\mu \right)^{\frac{1}{s}} \left(\int_X |\nabla_H f_n - \nabla_H^{\mu,q} f|_H^q d\mu \right)^{\frac{p}{q}} \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

So $\lim_{n \rightarrow +\infty} \nabla_H f_n = \nabla_H^{\mu,q} f$ in $L^p(X, \nu; H)$. The furthermore part is now obvious. \square

It is important to observe some basic properties of the space $W^{1,p}(X, \nu)$.

Proposition 4.5. *Let $p \geq \frac{t}{t-s'}$. The following holds:*

- (1) $W^{1,p}(X, \nu)$ is reflexive;

- (2) for every $q \in (ps', +\infty)$, $W^{1,q}(X, \mu) \hookrightarrow W^{1,p}(X, \nu)$. In particular if G satisfies Hypothesis 1.2 then $G \in W^{1,b}(X, \nu)$ for every $b \geq \frac{t}{t-s'}$. The same is true for the spaces $W^{1,q}(X, \mu; H)$ and $W^{1,p}(X, \nu; H)$;
- (3) if $\int_X w^{r(r-1)^{-1}} d\mu < +\infty$ for some $r \in (0, 1)$, then $W^{1,p}(X, \nu) \hookrightarrow W^{1,pr}(X, \mu)$. The same is true for the spaces $W^{1,q}(X, \mu; H)$ and $W^{1,p}(X, \nu; H)$;
- (4) let $(\varphi_n)_{n \in \mathbb{N}} \in W^{1,p}(X, \nu)$. If φ_n converges pointwise ν -a.e. to φ and

$$\sup_{n \in \mathbb{N}} \|\varphi_n\|_{W^{1,p}(X, \nu)} < +\infty,$$

then $\varphi \in W^{1,p}(X, \nu)$;

- (5) let $q \geq \frac{t}{t-s'}$, $\varphi \in W^{1,p}(X, \nu)$ and $\psi \in W^{1,q}(X, \nu)$. If $\frac{pq}{p+q} \geq \frac{t}{t-s'}$ then

$$\varphi\psi \in W^{1,r}(X, \nu) \quad \text{for every } r \in \left[\frac{t}{t-s'}, \frac{pq}{p+q} \right],$$

$$\text{and } \nabla_H(\varphi\psi) = \varphi \nabla_H \psi + \psi \nabla_H \varphi;$$

Proof. (1) It is easily seen that $L^p(X, \nu) \times L^p(X, \nu; H)$ is a reflexive Banach space when endowed with the norm

$$\|(f, \Phi)\| = \|f\|_{L^p(X, \nu)} + \|\Phi\|_{L^p(X, \nu; H)}.$$

This is due to the fact that the weakly closed and norm bounded sets of both $L^p(X, \nu)$ and $L^p(X, \nu; H)$ are weakly compact (see [DU77, Corollary IV.1.2]). Thus the set

$$\{(f, \Phi) \in L^p(X, \nu) \times L^p(X, \nu; H) \mid \|(f, \Phi)\| \leq 1\}$$

is weakly compact and so $L^p(X, \nu) \times L^p(X, \nu; H)$ is reflexive. The operator $T : W^{1,p}(X, \nu) \rightarrow L^p(X, \nu) \times L^p(X, \nu; H)$ defined as

$$T(f) = (f, \nabla_H f),$$

is an isometric embedding, which implies that the range of T is closed $L^p(X, \nu) \times L^p(X, \nu; H)$. Thus $T(W^{1,p}(X, \nu))$ is reflexive, being a closed subspace of a reflexive space. So $W^{1,p}(X, \nu)$ is reflexive too, being isometric to a reflexive space.

- (2) Let $f \in W^{1,q}(X, \mu)$. Using the Hölder inequality we get

$$\int_X |f|^p d\nu = \int_X |f|^p w d\mu \leq \left(\int_X |f|^{ps'} d\mu \right)^{\frac{1}{s'}} \left(\int_X w^s d\mu \right)^{\frac{1}{s}},$$

and the right hand side is finite whenever $ps' < q$. Using Proposition 4.4, the same inequality holds for $\nabla_H f$. Thus the statement follows.

- (3) Let $f \in W^{1,p}(X, \nu)$. Using the Hölder inequality we get

$$\int_X |f|^{pr} d\mu = \int_X \frac{|f|^{pr}}{w} d\nu \leq \left(\int_X |f|^p d\nu \right)^r \left(\int_X w^{\frac{r}{r-1}} d\mu \right)^{1-r}.$$

Using the same argument in Proposition 4.4 it is possible to prove the same inequality for $\nabla_H f$. Thus the statement follows.

- (4) The statement is a consequence of the Banach–Saks property, i.e. every bounded sequence $(\varphi_n)_{n \in \mathbb{N}} \subseteq L^p(X, \nu; H)$ has a subsequence $(\varphi_{n_k})_{k \in \mathbb{N}}$ such that the sequence

$$\left(\frac{\varphi_{n_1} + \cdots + \varphi_{n_k}}{k} \right)_{k \in \mathbb{N}}$$

strongly converges in $L^p(X, \nu; H)$. The proof is the same of [Bog98, Lemma 5.4.4] with obvious adjustments.

- (5) Standard calculations. □

Proposition 4.6. *Let $p \geq \frac{t}{t-s'}$ and $\theta \in \mathcal{C}_b^1(\mathbb{R})$. If $\varphi \in W^{1,p}(X, \nu)$, then $\theta \circ \varphi \in W^{1,p}(X, \nu)$ and*

$$\nabla_H(\theta \circ \varphi) = (\theta' \circ \varphi) \nabla_H \varphi \quad \nu\text{-a.e.}$$

Proof. Let $(\varphi_n)_{n \in \mathbb{N}} \subseteq \mathcal{FC}_b^\infty(X)$ be such that φ_n converges in $W^{1,p}(X, \nu)$ and pointwise ν -a.e. to φ . Applying (4) of Proposition 4.5 to the sequence $\theta \circ \varphi_n$ we get $\theta \circ \varphi \in W^{1,p}(X, \nu)$. Furthermore we know that, for every $n \in \mathbb{N}$, $\theta \circ \varphi_n \in \mathcal{FC}_b^\infty(X)$ and

$$\nabla_H(\theta \circ \varphi_n) = (\theta' \circ \varphi_n) \nabla_H \varphi_n.$$

By Proposition 4.2 and the fact that $\theta \in \mathcal{C}_b^1(\mathbb{R})$, we can let $n \rightarrow +\infty$ and get the statement. □

Proposition 4.7. *Let $p \geq \frac{t}{t-s'}$ and $u \in W^{1,p}(X, \nu)$. Then $|u| \in W^{1,p}(X, \nu)$ and*

$$\nabla_H |u| = \text{sign}(u) \nabla_H u.$$

Furthermore $\nabla_H u(x) = 0$ ν -a.e. in $\{y \in X \mid u(y) = 0\}$.

Proof. The proof is the same as in [DPL14, Lemma 2.7], we just use Lemma 4.1 instead of the classical integration by parts formula for the Gaussian measure (Equation 2.1). □

5. DIVERGENCE OPERATOR

To have a self-contained paper, we introduce the weighted divergence operator.

Definition 5.1. Let $\Phi \in L^1(X, \nu; X)$ be a vector field. We say that Φ admits *divergence* if a function $g \in L^1(X, \nu)$ exists such that

$$(5.1) \quad \int_X \partial_\Phi f d\nu = - \int_X f g d\nu \quad \text{for every } f \in \mathcal{FC}_b^\infty(X);$$

where $(\partial_\Phi f)(x)$ is defined in Equation (2.2). If such a function exists then we let $\text{div}_\nu \Phi := g$. Observe that, when $\text{div}_\nu \Phi$ exists, it is unique by Proposition 3.6. We denote by $D(\text{div}_\nu)$ the domain of div_ν in $L^1(X, \nu; X)$. Lastly observe that if $\Phi \in L^1(X, \nu; H)$, then $(\partial_\Phi f)(x) = \langle \nabla_H f(x), \Phi(x) \rangle_H$. In this case Equation (5.1) becomes

$$(5.2) \quad \int_X \langle \nabla_H f(x), \Phi(x) \rangle_H d\nu = - \int_X f g d\nu \quad \text{for every } f \in \mathcal{FC}_b^\infty(X).$$

We are now interested in conditions ensuring the L^p -integrability of $\text{div}_\nu v$ and non emptiness of $D(\text{div}_\nu)$. We start by studying the case of a vector field in $W^{1,2}(X, \nu; H)$.

Proposition 5.2. *If $\log w \in W^{2,t}(X, \mu)$, then for every $f, g \in \mathcal{FC}_b^\infty(X)$*

$$\begin{aligned} & \int_X \left(f\widehat{h} - f\partial_h \log w - \partial_h f \right) \left(g\widehat{k} - g\partial_k \log w - \partial_k g \right) d\nu = \\ & = \langle h, k \rangle_H \int_X f g d\nu - \int_X f g \partial_h \partial_k \log w d\nu + \int_X \partial_k f \partial_h g d\nu. \end{aligned}$$

Proof. By the integration by parts formula (Lemma 4.1) we get

$$\begin{aligned} & \int_X \left(f\widehat{h} - f\partial_h \log w - \partial_h f \right) \left(g\widehat{k} - g\partial_k \log w - \partial_k g \right) d\nu = \\ & = \int_X f \partial_h (g\widehat{k}) d\nu - \int_X f \partial_h (g \partial_k \log w) d\nu - \int_X f \partial_h \partial_k g d\nu. \end{aligned}$$

Recalling that $\partial_h \widehat{k} = \langle h, k \rangle_H$ we get

$$\begin{aligned} & \int_X \left(f\widehat{h} - f\partial_h \log w - \partial_h f \right) \left(g\widehat{k} - g\partial_k \log w - \partial_k g \right) d\nu = \\ & = \langle h, k \rangle_H \int_X f g d\nu - \int_X f g \partial_h \partial_k \log w d\nu + \int_X \partial_k f \partial_h g d\nu. \end{aligned}$$

□

Proposition 5.3. *Let $\log w \in W^{2,t}(X, \mu)$, for some $t \geq 2s'$, and assume that $C \geq 0$ exists such that for every $(\xi_i)_{i \in \mathbb{N}} \in \ell_2$ and for μ -a.e. $x \in X$ we have*

$$(5.3) \quad \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} (\delta_{ij} - \partial_i \partial_j \log w(x)) \xi_i \xi_j \leq C \sum_{i=1}^{+\infty} \xi_i^2 \quad \mu\text{-a.e.}$$

where $(\xi_i)_{i \in \mathbb{N}} \in \ell_2$ μ -a.e. for every $i \in \mathbb{N}$. Then every field $\Phi \in W^{1,2}(X, \nu; H)$ has a divergence $\operatorname{div}_\nu \Phi \in L^2(X, \nu)$ and for every $f \in W^{1,2}(X, \nu)$, the following equality holds:

$$\int_X \langle \nabla_H f(x), \Phi(x) \rangle_H d\nu(x) = - \int_X f(x) \operatorname{div}_\nu \Phi(x) d\nu(x).$$

Furthermore, if $\varphi_n = \langle \Phi, e_n \rangle_H$ for every $n \in \mathbb{N}$ where $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of H , then

$$\operatorname{div}_\nu \Phi(x) = \sum_{n=1}^{+\infty} (\partial_n \varphi_n(x) + \varphi_n(x) \partial_n \log w(x) - \varphi_n(x) \widehat{e}_n),$$

and $\|\operatorname{div}_\nu \Phi\|_{L^2(X, \nu)} \leq \max\{\sqrt{C}, 1\} \|\Phi\|_{W^{1,2}(X, \nu; H)}.$

Proof. Let $\Phi(x) = \sum_{i=1}^n \varphi_i(x) e_i$ with $\varphi_i \in \mathcal{FC}_b^\infty(X)$ for every $i = 1, \dots, n$. Then by the integration by parts formula (Lemma 4.1) if $f \in \mathcal{FC}_b^\infty(X)$ we have

$$\int_X \langle \nabla_H f(x), \Phi(x) \rangle_H d\nu(x) = - \int_X f(x) \left(\sum_{i=1}^n (\partial_i \varphi_i(x) + \varphi_i(x) \partial_i \log w(x) - \varphi_i(x) \widehat{e}_i) \right) d\nu(x).$$

Put

$$\operatorname{div}_\nu \Phi(x) = \sum_{i=1}^n (\partial_i \varphi_i(x) + \varphi_i(x) \partial_i \log w(x) - \varphi_i(x) \widehat{e}_i).$$

By Proposition 5.2

$$\begin{aligned}
\int_X (\operatorname{div}_\nu \Phi)^2 d\nu &= \sum_{i=1}^n \int_X |\varphi_i|^2 d\nu + \sum_{i=1}^n \sum_{j=1}^n \int_X \partial_j \varphi_i \partial_i \varphi_j d\nu - \sum_{i=1}^n \sum_{j=1}^n \int_X \varphi_i \varphi_j \partial_i \partial_j \log w d\nu = \\
(5.4) \quad &= \int_X \sum_{i=1}^n \sum_{j=1}^n (\delta_{ij} - \partial_i \partial_j \log w) \varphi_i \varphi_j d\nu + \int_X \operatorname{trace}_H((\nabla_H \Phi(x))^2) d\nu(x) \leq \\
&\leq C \|\Phi\|_{L^2(X, \nu; H)}^2 + \int_X \|\nabla_H \Phi(x)\|_{\mathcal{H}}^2 d\nu(x) \leq \max\{C, 1\} \|\Phi\|_{W^{1,2}(X, \nu; H)}^2.
\end{aligned}$$

Let $(\Phi^n)_{n \in \mathbb{N}}$ be a sequence of fields as above which converges to Φ in $W^{1,2}(X, \nu; H)$. The sequence $(\operatorname{div}_\nu \Phi^n)$ is Cauchy in $L^2(X, \nu)$ and, hence it converges to some element, which is the candidate to be $\operatorname{div}_\nu \Phi$. By the integration by parts formula (Lemma 4.1) it is easily seen that such an element satisfies (5.2). Finally, the fact that (5.2) holds also for every $f \in W^{1,2}(X, \nu)$, follows by a standard approximation argument. \square

Observe that condition (5.3) is satisfied whenever $\nabla_H^2 \log w$ is bounded from below μ -a.e., in particular when $\log w$ is a convex function.

The following example show that Condition (5.3) is not necessary for the domain of the divergence to be not empty.

Example 5.4. Let X be a separable Hilbert space, with norm $\|\cdot\|_X$ and inner product $(\cdot, \cdot)_X$, endowed with a nondegenerate centered Gaussian measure μ , with covariance Q . We fix an orthonormal basis $(v_n)_{n \in \mathbb{N}}$ of X of eigenvectors of Q , i.e. $Qv_k = \lambda_k v_k$, and the corresponding orthonormal basis of $H = Q^{\frac{1}{2}}(X)$ is $(e_n := \sqrt{\lambda_n} v_n)_{n \in \mathbb{N}}$. Recall that $\hat{e}_n(x) = \frac{(x, v_n)_X}{\sqrt{\lambda_n}}$. In this example we take $\lambda_n = 4^{-n}$.

Let $w(x) := (x, x)_X^2 = \sum_{n=1}^{+\infty} (x, v_n)_X^2 = \sum_{n=1}^{+\infty} \lambda_n^{-1} (x, e_n)_X^2$. Easy calculations give

$$\begin{aligned}
\partial_i w(x) &= 4(x, x)_X (x, e_i)_X, \quad \partial_i \log w(x) = 4 \frac{(x, e_i)_X}{(x, x)_X}, \\
\partial_i \partial_j \log w(x) &= 4 \left(\frac{(e_i, e_j)_X}{(x, x)_X} - \frac{2(x, e_i)_X (x, e_j)_X}{(x, x)_X^2} \right).
\end{aligned}$$

We get $w \in W^{1,s}(X, \mu)$ and $\log w \in W^{2,t}(X, \mu)$ for every $t, s > 1$. Therefore $W^{1,p}(X, \nu)$ is well defined for every $p > 1$ (see Definition 4.3).

We want to show that w does not satisfy Condition (5.3). For every $(\xi_i)_{i \in \mathbb{N}} \in \ell_2$ we have

$$\begin{aligned}
(5.5) \quad &\sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} (\delta_{ij} - \partial_i \partial_j \log w(x)) \xi_i \xi_j = \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} \left(\delta_{ij} - 4 \frac{(e_i, e_j)_X}{(x, x)_X} + 8 \frac{(x, e_i)_X (x, e_j)_X}{(x, x)_X^2} \right) \xi_i \xi_j = \\
&= \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} \left(\delta_{ij} - 4 \delta_{ij} \frac{\sqrt{\lambda_i \lambda_j}}{(x, x)_X} + 8 \frac{\sqrt{\lambda_i \lambda_j} (x, v_i)_X (x, v_j)_X}{(x, x)_X^2} \right) \xi_i \xi_j.
\end{aligned}$$

By contradiction assume that $C \geq 0$ exists such that Equation (5.3) holds. Choosing $\xi_i = (x, v_i)$ for every fixed $x \in X$, we get

$$\begin{aligned} & \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} (\delta_{ij} - \partial_i \partial_j \log w(x)) \xi_i \xi_j = \\ &= \sum_{i=1}^{+\infty} (x, v_i)_X^2 - \frac{4}{(x, x)_X} \sum_{i=1}^{+\infty} \frac{(x, v_i)_X^2}{4^i} + \frac{8}{(x, x)_X^2} \left(\sum_{i=1}^{+\infty} \frac{(x, v_i)_X^2}{2^i} \right)^2 \leq C \sum_{i=1}^{+\infty} (x, v_i)_X^2 \quad \mu\text{-a.e..} \end{aligned}$$

Let $A = \left\{ x \in X \mid \|x\|_X^2 < \sqrt{2^{-1}} \sum_{i=1}^{+\infty} 4^{-i} (x, v_i)_X^2 \right\}$ and $B_r = \{x \in X \mid \|x\|_X < r\}$ for $0 < r < 1$. Observe that $A \cap B_r$ are open non-empty subsets of X for every $r \in (0, 1)$, this means that $\mu(A \cap B_r) > 0$. Fix $x \in A \cap B_r$ for some $r \in (0, 1)$, then

$$\begin{aligned} & (C-1) \sum_{i=1}^{+\infty} (x, v_i)_X^2 + \frac{4}{(x, x)_X} \sum_{i=1}^{+\infty} \frac{(x, v_i)_X^2}{4^i} - \frac{8}{(x, x)_X^2} \left(\sum_{i=1}^{+\infty} \frac{(x, v_i)_X^2}{2^i} \right)^2 \leq \\ & \leq (C-1) \sum_{i=1}^{+\infty} (x, v_i)_X^2 + \frac{4}{(x, x)_X} \sum_{i=1}^{+\infty} \frac{(x, v_i)_X^2}{4^i} - \frac{8}{(x, x)_X^2} \left(\sum_{i=1}^{+\infty} \frac{(x, v_i)_X^2}{4^i} \right)^2 \leq \\ (5.6) \quad & \leq (C-1) \|x\|_X^2 + 4 - 2 \|x\|_X^4 \frac{8}{\|x\|_X^4} = (C-1) \|x\|_X^2 - 12 \end{aligned}$$

Observe that if $C > 1$ and $A \cap B_{(6(C-1)^{-1})^{\frac{1}{2}}}$, then (5.6) ≤ -6 μ -a.e., a contradiction. Thus our example does not satisfy Equation (5.3).

Observe that it is possible to define the divergence operator for every $\Phi \in W^{1,2\alpha}(X, \mu; H)$ with $\alpha > 1$. Indeed let $\varphi_n = \langle \Phi, e_n \rangle_H$ for every $n \in \mathbb{N}$, then by Equation (5.5)

$$\begin{aligned} & \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} (\delta_{ij} - \partial_i \partial_j \log w(x)) \varphi_i \varphi_j = \\ &= \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} \left(\delta_{ij} - 4 \delta_{ij} \frac{\sqrt{\lambda_i \lambda_j}}{(x, x)_X} + 8 \frac{\sqrt{\lambda_i \lambda_j} (x, v_i)_X (x, v_j)_X}{(x, x)_X^2} \right) \varphi_i \varphi_j = \\ &= \sum_{i=1}^{+\infty} \varphi_i^2 - 4 \sum_{i=1}^{+\infty} \frac{\lambda_i \varphi_i^2}{(x, x)_X} + \frac{8}{(x, x)_X^2} \left(\sum_{i=1}^{+\infty} \sqrt{\lambda_i} \varphi_i (x, v_i)_X \right)^2 \leq \\ &\leq |\Phi|_H^2 + 8 \frac{(x, \sum_{i=1}^{+\infty} \varphi_i e_i)_X^2}{(x, x)_X^2} = |\Phi|_H^2 + 8 \frac{(x, \Phi)_X^2}{(x, x)_X^2} \leq (*) |\Phi|_H^2 + 8 \frac{\|\Phi\|_X^2}{(x, x)_X} \leq |\Phi|_H^2 + 8K \frac{|\Phi|_H^2}{(x, x)_X}, \end{aligned}$$

where (*) follows by the Schwarz inequality and $\|h\|_X \leq K \|h\|_H$ for every $h \in H$. We can now integrate both terms of the inequality

$$\begin{aligned} & \int_X \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} (\delta_{ij} - \partial_i \partial_j \log w(x)) \varphi_i \varphi_j d\nu \leq \int_X \left(|\Phi|_H^2 + 8K \frac{|\Phi|_H^2}{(x, x)_X} \right) d\nu = \\ &= \int_X |\Phi|_H^2 (x, x)_X^2 d\nu + 8K \int_X |\Phi|_H^2 (x, x)_X d\nu. \end{aligned}$$

By the Hölder inequality and Fernique's theorem (see [Bog98, Theorem 2.8.5]) there exists a constant \overline{K} such that

$$\int_X \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} (\delta_{ij} - \partial_i \partial_j \log w(x)) \varphi_i \varphi_j d\nu \leq \overline{K} \|\Phi\|_{L^{2\alpha}(X, \mu; H)}^2$$

Fix now a field of the following form: $\Phi(x) = \sum_{i=1}^n \varphi_i(x) e_i$ with $\varphi_i \in \mathcal{FC}_b^\infty(X)$ for every $i = 1, \dots, n$. Following the same steps of the proof of Proposition 5.3 we rewrite Equation (5.4) as

$$\int_X (\operatorname{div}_\nu \Phi)^2 d\nu \leq \overline{K} \|\Phi\|_{L^{2\alpha}(X, \mu; H)}^2 + \int_X \|\nabla_H \Phi(x)\|_{\mathcal{H}}^2 d\nu(x) \leq \max\{\overline{K}, 1\} \|\Phi\|_{W^{1,2\alpha}(X, \mu; H)}^2.$$

By the same argument used at the end of the proof of Proposition 5.3 and by (2) of Proposition 4.5 (since it implies that $W^{1,2\alpha}(X, \mu; H) \subseteq L^1(X, \nu; X)$ for every $\alpha > 1$), we obtain that $W^{1,2\alpha}(X, \mu; H) \subseteq D(\operatorname{div}_\nu)$ for every $\alpha > 1$.

Since in several practical examples the vector field we are interested in belongs to $W^{1,q}(X, \mu; H)$ for some $q > 1$, the following proposition might be useful.

Proposition 5.5. *Let $\Phi \in W^{1,q}(X, \mu; H)$ for some $q \geq \frac{s't}{t-s'}$. Then $\operatorname{div}_\nu \Phi$ exists and it belongs to $L^p(X, \nu)$ for every $p \in \left[1, \frac{qt}{s'(q+t)}\right]$. Furthermore*

$$(5.7) \quad \operatorname{div}_\nu \Phi = \operatorname{div}_\mu \Phi + \langle \Phi, \nabla_H \log w \rangle_H$$

and $\|\operatorname{div}_\nu \Phi\|_{L^p(X, \nu)} \leq C_p \|\Phi\|_{W^{1,q}(X, \mu; H)}$ for some $C_p > 0$. Finally if $\Phi \in W^{1,q}(X, \mu; H)$ for every $q > 1$, then $\operatorname{div}_\nu \Phi \in L^{\frac{t}{s'}}(X, \nu)$.

Proof. By the Hölder inequality $\langle \Phi, \nabla_H \log w \rangle_H \in L^{\frac{qt}{q+t}}(X, \mu)$ and by [Bog98, Proposition 5.8.8] $\operatorname{div}_\mu \Phi \in L^q(X, \mu)$. By (2) of Proposition 4.5 both $\operatorname{div}_\mu \Phi$ and $\langle \Phi, \nabla_H \log w \rangle_H$ belong to $L^p(X, \nu)$ for every $p \in \left[1, \frac{qt}{s'(q+t)}\right]$. Let $f \in \mathcal{FC}_b^\infty(X)$ and $(w_n)_{n \in \mathbb{N}} \in \mathcal{FC}_b^\infty(X)$ be a sequence of positive functions which converges to w pointwise and in $W^{1,s}(X, \mu)$ (so that also $\log w_n$ converges to $\log w$ both pointwise and in $W^{1,t}(X, \mu)$), then

$$\begin{aligned} \int_X f(\operatorname{div}_\mu \Phi + \langle \Phi, \nabla_H \log w_n \rangle_H) w_n d\mu &= \int_X (f w_n) \operatorname{div}_\mu \Phi d\mu + \int_X f \langle \Phi, \nabla_H \log w_n \rangle_H w_n d\mu = \\ &= - \int_X \langle \nabla_H(f w_n), \Phi \rangle d\mu + \int_X f \langle \Phi, \nabla_H \log w_n \rangle_H w_n d\mu = \\ &= - \int_X \langle \nabla_H f, \Phi \rangle w_n d\mu - \int_X f \langle \nabla_H(w_n), \Phi \rangle d\mu + \int_X f \langle \Phi, \nabla_H \log w_n \rangle_H w_n d\mu = \\ &= - \int_X \langle \nabla_H f, \Phi \rangle w_n d\mu - \int_X f \langle \nabla_H(\log w_n), \Phi \rangle w_n d\mu + \int_X f \langle \Phi, \nabla_H \log w_n \rangle_H w_n d\mu = \\ &= - \int_X \langle \nabla_H f, \Phi \rangle w_n d\mu. \end{aligned}$$

Letting $n \rightarrow +\infty$ we obtain that $\operatorname{div}_\nu \Phi = \operatorname{div}_\mu \Phi + \langle \Phi, \nabla_H \log w \rangle_H$. The inequality

$$\|\operatorname{div}_\nu \Phi\|_{L^p(X, \nu)} \leq C_p \|\Phi\|_{W^{1,q}(X, \mu; H)}$$

follows by a standard Hölder inequality application and [Bog98, Proposition 5.8.8]. \square

6. SOBOLEV SPACES ON SUBLEVEL SETS

Let G be a function satisfying Hypothesis 1.2. We are interested in Sobolev spaces on sets of the form $G^{-1}(-\infty, 0)$. By Corollary 3.5, $\text{Lip}(G^{-1}(-\infty, 0))$ is a dense subspace in $L^p(G^{-1}(-\infty, 0), \nu)$. Whenever $p \geq \frac{t}{t-s'}$ it is possible to define

$$\nabla_H^0 : \text{Lip}(G^{-1}(-\infty, 0)) \longrightarrow L^p(G^{-1}(-\infty, 0), \nu; H)$$

in the following way: for every $\varphi \in \text{Lip}(G^{-1}(-\infty, 0))$, with Lipschitz constant L_φ , consider the McShane extension (see [McS34])

$$\varphi_M(x) = \sup \{ \varphi(y) + L_\varphi \|x - y\|_X \mid G(y) < 0 \}.$$

It is well known that $\varphi_M \in \text{Lip}(X) \subseteq W^{1,q}(X, \nu)$ for some $q \geq \frac{t}{t-s'}$ ((2) of Proposition 4.5 and [Bog98, Theorem 5.11.2]). Let

$$\nabla_H^0 \varphi := \nabla_H \varphi_M.$$

Proposition 6.1. *Let $p \geq \frac{t}{t-s'}$. The operator*

$$\nabla_H^0 : \text{Lip}(G^{-1}(-\infty, 0)) \rightarrow L^p(G^{-1}(-\infty, 0), \nu; H)$$

is closable in $L^p(G^{-1}(-\infty, 0), \nu)$.

Proof. Let $(f_k)_{k \in \mathbb{N}} \subseteq \text{Lip}(G^{-1}(-\infty, 0))$ such that

$$\begin{aligned} \lim_{k \rightarrow +\infty} f_k &= 0 && \text{in } L^p(G^{-1}(-\infty, 0), \nu); \\ \lim_{k \rightarrow +\infty} \nabla_H^0 f_k &= \Phi && \text{in } L^p(G^{-1}(-\infty, 0), \nu; H). \end{aligned}$$

We want to prove that

$$\int_{G^{-1}(-\infty, 0)} \langle \Phi(x), e_i \rangle u(x) d\nu(x) = 0$$

for every $i \in \mathbb{N}$ and $u \in \mathcal{FC}_b^\infty(X)$. Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ a smooth function such that $\|\eta\|_\infty \leq 1$, $\|\eta'\|_\infty \leq 1$ and

$$\eta(\xi) = \begin{cases} 0 & \xi \geq -1 \\ 1 & \xi \leq -2 \end{cases}$$

Let $\eta_n(\xi) := \eta(n\xi)$ and $u_n(x) = u(x)\eta_n(G(x))$. Observe that u_n converges pointwise ν -a.e. to u and $|u_n| \leq |u|$ ν -a.e., then by Lebesgue's dominated convergence theorem

$$\lim_{n \rightarrow +\infty} \int_{G^{-1}(-\infty, 0)} \langle \Phi(x), e_i \rangle u_n(x) d\nu(x) = \int_{G^{-1}(-\infty, 0)} \langle \Phi(x), e_i \rangle u(x) d\nu(x).$$

By (5) of Proposition 4.5, for every $n \in \mathbb{N}$ we have $u_n \in W^{1,r}(X, \nu)$, for every $r \geq \frac{t}{t-s'}$, and by Proposition 4.6

$$\partial_i u_n(x) = \partial_i u(x) \eta_n(G(x)) + u(x) \eta'_n(G(x)) \partial_i G(x).$$

Observe that

$$\int_X u_n \partial_i f_k d\nu = \int_X f_k u_n (\hat{e}_i - \partial_i \log w) d\nu - \int_X f_k \partial_i u (\eta_n \circ G) d\nu - \int_X f_k u (\eta' \circ G) \partial_i G d\nu,$$

and the following estimates holds:

$$\begin{aligned} \int_X |\partial_i f_k u_n - \langle \Phi, e_i \rangle_H u| d\nu &\leq \int_X |\partial_i f_k| |u_n - u| d\nu + \int_X |\partial_i f_k - \langle \Phi, e_i \rangle_H| |u| d\nu \leq \\ &\leq \left(\int_X |\partial_i f_k|^p d\nu \right)^{\frac{1}{p}} \left(\int_X |u_n - u|^{p'} d\nu \right)^{\frac{1}{p'}} + \left(\int_X |\partial_i f_k - \langle \Phi, e_i \rangle_H|^p d\nu \right)^{\frac{1}{p}} \left(\int_X |u|^{p'} d\nu \right)^{\frac{1}{p'}}, \end{aligned}$$

this means $\lim_{n \rightarrow +\infty} \lim_{k \rightarrow +\infty} \int_X u_n \partial_i f_k d\nu = \int_{G^{-1}(-\infty, 0)} \langle \Phi, e_i \rangle_H u d\nu$. Furthermore for every $n \in \mathbb{N}$ we get

$$\begin{aligned} \int_X |f_k \partial_i u(\eta_n \circ G)| d\nu &\leq \int_X |f_k \partial_i u| d\nu \leq \left(\int_X |f_k|^p d\nu \right)^{\frac{1}{p}} \left(\int_X |\partial_i u|^{p'} d\nu \right)^{\frac{1}{p'}} \xrightarrow{k \rightarrow +\infty} 0; \\ \int_X |f_k u(\eta'_n \circ G) \partial_i G| d\nu &\leq \|u\|_\infty \left(\int_X |f_k|^p d\nu \right)^{\frac{1}{p}} \left(\int_X |\partial_i G|^{p's'} d\mu \right)^{\frac{1}{p's'}} \left(\int_X w^s d\mu \right)^{\frac{1}{p's'}} \xrightarrow{k \rightarrow +\infty} 0, \end{aligned}$$

where the last limit follows by to Hypothesis 1.2;

$$\begin{aligned} \int_X |f_k u_n \widehat{e}_i| d\nu &\leq \|u\|_\infty \left(\int_X |f_k|^p d\nu \right)^{\frac{1}{p}} \left(\int_X |\widehat{e}_i|^{p'} d\nu \right)^{\frac{1}{p'}} \xrightarrow{k \rightarrow +\infty} 0; \\ \int_X |f_k u_n \partial_i \log w| d\nu &\leq \|u\|_\infty \left(\int_X |f_k|^p d\nu \right)^{\frac{1}{p}} \left(\int_X |w|^s d\mu \right)^{\frac{1}{p's'}} \left(\int_X |\partial_i \log w|^{p's'} d\mu \right)^{\frac{1}{p's'}} \xrightarrow{k \rightarrow +\infty} 0, \end{aligned}$$

and the last limit exists whenever $p's' \leq t$. \square

Definition 6.2 (Weighted Sobolev space on sublevel sets). Let $p \geq \frac{t}{t-s'}$. We denote by $W^{1,p}(G^{-1}(-\infty, 0), \nu)$ the domain of the closure of the operator ∇_H^0 (which we will denote by the symbol ∇_H) in $L^p(G^{-1}(-\infty, 0), \nu)$. It is a Banach space with the graph norm

$$\|f\|_{W^{1,p}(G^{-1}(-\infty, 0), \nu)} = \left(\int_{G^{-1}(-\infty, 0)} |f(x)|^p d\nu(x) \right)^{\frac{1}{p}} + \left(\int_{G^{-1}(-\infty, 0)} |\nabla_H f(x)|^p_H d\nu(x) \right)^{\frac{1}{p}}.$$

In order to proceed we need to recall the definition of the Feyel–de La Pradelle Hausdorff–Gauss surface measure. If $m \geq 2$ and $F = \mathbb{R}^m$ equipped with a norm $\|\cdot\|$, we define

$$d\theta^F(x) = \frac{1}{(2\pi)^{\frac{m}{2}}} e^{-\frac{\|x\|^2}{2}} dH_{m-1}(x),$$

where H_{m-1} is the spherical $(m-1)$ -dimensional Hausdorff measure in \mathbb{R}^m , i.e.

$$H_{m-1}(A) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{n \in \mathbb{N}} w_{m-1} r_n^{m-1} \left| A \subseteq \bigcup_{n \in \mathbb{N}} B(x_n, r_n), r_n < \delta, \text{ for every } n \in \mathbb{N} \right| \right\},$$

where $w_{m-1} = \pi^{\frac{m-1}{2}} (\Gamma(\frac{m+1}{2}))^{-1}$. For every m -dimensional $F \subseteq H$ we consider the orthogonal projection (along H) on F :

$$x \mapsto \sum_{n=1}^m \langle x, f_n \rangle_H f_n \quad x \in H$$

where $\{f_n\}_{n=1}^m$ is an orthonormal basis of F . There exists a μ -measurable projection π^F on F , defined in the whole X , that extends it (see [Bog98, Theorem 2.10.11]). We denote with $\tilde{F} := \ker \pi^F$ and with $\mu_{\tilde{F}}$ the image of the measure μ on \tilde{F} through $I - \pi^F$. Finally we denote with μ_F the image of the measure μ on F through π^F , which is the standard Gaussian measure on \mathbb{R}^m if we identify F with \mathbb{R}^m . Let $A \subseteq X$ a Borel set and identify F with \mathbb{R}^m , we set

$$\rho^F(A) := \int_{\ker \pi^F} \theta^F(A_x) d\mu_{\tilde{F}}(x),$$

where $A_x = \{y \in F \mid x + y \in A\}$. The map $F \mapsto \rho^F(A)$ is well defined and increasing, namely if $F_1 \subseteq F_2$, then $\rho^{F_1}(A) \leq \rho^{F_2}(A)$ (see [AMP10, Lemma 3.1] and [Fey01, Proposition 3.2]). The Feyel-de La Pradelle Hausdorff–Gauss surface measure is defined by

$$\rho(A) = \sup \{ \rho^F(A) \mid F \subseteq H, F \text{ is finite dimensional} \}.$$

Recall that if $C_{1,p}(A) = 0$ for a Borel set $A \subseteq X$ and some $p > 1$, then $\rho^F(A) = 0$ for every $F \subseteq H$ finite dimensional. In particular $\rho(A) = 0$ (see [FdLP91, Theorem 9]).

The following equalities are what will allow us to define the trace operator in the following section.

Proposition 6.3. *Let $k \in \mathbb{N}$. Then for every $\varphi \in \text{Lip}_b(G^{-1}(-\infty, 0))$ and G satisfying Hypothesis 1.2, we have*

$$\int_{G^{-1}(-\infty, 0)} (\partial_k \varphi + \varphi \partial_k \log w - \varphi \widehat{e}_k) d\nu = \int_{G^{-1}(0)} \left(\frac{\varphi \partial_k G}{|\nabla_H G|_H} \right)_{|_{G^{-1}(0)}} w d\rho.$$

Proof. The proof is the same as [CL14, Equation (1.1), proof in Proposition 4.1]. \square

Proposition 6.4. *Let $q \geq 1$. Then for every $\varphi \in \text{Lip}_b(G^{-1}(-\infty, 0))$ and G satisfying Hypothesis 1.2, we have*

$$\int_{G^{-1}(-\infty, 0)} (q\varphi|\varphi|^{q-2} \langle \nabla_H \varphi, \nabla_H G \rangle_H - |\varphi|^q \text{div}_\nu \nabla_H G) d\nu = \int_{G^{-1}(0)} (|\varphi|^q |\nabla_H G|_H)_{|_{G^{-1}(0)}} w d\rho,$$

and

$$\int_{G^{-1}(-\infty, 0)} \left(q\varphi|\varphi|^{q-2} \left\langle \nabla_H \varphi, \frac{\nabla_H G}{|\nabla_H G|_H} \right\rangle + \text{div}_\nu \frac{\nabla_H G}{|\nabla_H G|_H} |\varphi|^q \right) d\nu = \int_{G^{-1}(0)} (|\varphi|^q)_{|_{G^{-1}(0)}} w d\rho.$$

Proof. The proof is the same as [CL14, Proposition 4.1]. \square

7. TRACES OF SOBOLEV FUNCTIONS ON SUBLEVEL SETS

Throughout this section we will denote by G a function satisfying Hypothesis 1.2. The following result is fundamental for the definition of the trace operator.

Proposition 7.1. *Let $p \geq \frac{t}{t-s'}$. The following holds:*

- (1) *if $(\varphi_n)_{n \in \mathbb{N}} \subseteq \text{Lip}_b(G^{-1}(-\infty, 0))$ is a Cauchy sequence in $W^{1,p}(G^{-1}(-\infty, 0), \nu)$, then $(\varphi_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^q(G^{-1}(0), w\rho)$ for every $1 \leq q \leq p \frac{t-s'}{t}$;*

- (2) if $(\varphi_n)_{n \in \mathbb{N}}, (\psi_n)_{n \in \mathbb{N}} \subseteq \text{Lip}_b(G^{-1}(-\infty, 0))$ converge to φ in $W^{1,p}(G^{-1}(-\infty, 0), \nu)$, then $(\varphi_n|_{G^{-1}(0)})$ and $(\psi_n|_{G^{-1}(0)})$ converge to the same element in $L^q(G^{-1}(0), w\rho)$ for every $1 \leq q \leq p \frac{t-s'}{t}$.

Proof. Let $(\varphi_n)_{n \in \mathbb{N}} \subseteq \text{Lip}_b(G^{-1}(-\infty, 0))$ be a Cauchy sequence in $W^{1,p}(G^{-1}(-\infty, 0), \nu)$. By Proposition 6.4 we have for every $q \in [1, p \frac{t-s'}{t}]$

$$\begin{aligned} \int_{G^{-1}(0)} |\varphi_n - \varphi_m|^q w d\rho &= \int_{G^{-1}(-\infty, 0)} \left(q |\varphi_n - \varphi_m|^q (\varphi_n - \varphi_m) \left\langle \nabla_H(\varphi_n - \varphi_m), \frac{\nabla_H G}{|\nabla_H G|_H} \right\rangle_H \right) d\nu + \\ &+ \int_{G^{-1}(-\infty, 0)} \left(\text{div}_\nu \frac{\nabla_H G}{|\nabla_H G|_H} |\varphi_n - \varphi_m|^q \right) d\nu \xrightarrow{n, m \rightarrow +\infty} 0. \end{aligned}$$

All the convergences are assured by Proposition 5.5. Thus (1) is proved.

Let $(\varphi_n)_{n \in \mathbb{N}}, (\psi_n)_{n \in \mathbb{N}} \subseteq \text{Lip}_b(G^{-1}(-\infty, 0))$ satisfying (2) and let φ_∞ the limit of $(\varphi_n|_{G^{-1}(0)})$ in $L^q(G^{-1}(0), w\rho)$ for every $1 \leq q \leq p \frac{t-s'}{t}$ (it exists by (1)). By Proposition 6.4 we have for every $q \in [1, p \frac{t-s'}{t}]$

$$\begin{aligned} \int_{G^{-1}(0)} |\psi_n - \varphi_\infty|^q w d\rho &\leq 2^{q-1} \left(\int_{G^{-1}(0)} |\psi_n - \varphi_n|^q w d\rho + \int_{G^{-1}(0)} |\varphi_n - \varphi_\infty|^q w d\rho \right) = \\ &= 2^{q-1} \left(\int_{G^{-1}(-\infty, 0)} \left(q |\psi_n - \varphi_n|^q (\psi_n - \varphi_n) \left\langle \nabla_H(\psi_n - \varphi_n), \frac{\nabla_H G}{|\nabla_H G|_H} \right\rangle_H \right) d\nu + \right. \\ &+ \left. \int_{G^{-1}(-\infty, 0)} \left(\text{div}_\nu \frac{\nabla_H G}{|\nabla_H G|_H} |\psi_n - \varphi_n|^q \right) d\nu + \int_{G^{-1}(0)} |\varphi_n - \varphi_\infty|^q w d\rho \right) \xrightarrow{n, m \rightarrow +\infty} 0. \end{aligned}$$

Thus (2) is proved. \square

We are now in the position to define traces.

Definition 7.2. Let $p \geq \frac{t}{t-s'}$. If $\varphi \in W^{1,p}(G^{-1}(-\infty, 0), \nu)$ we define the trace of φ on $G^{-1}(0)$ as follows:

$$\text{Tr}_{G^{-1}(0)} \varphi = \lim_{n \rightarrow +\infty} \varphi_n|_{G^{-1}(0)} \quad \text{in } L^q(G^{-1}(0), w\rho) \text{ for every } q \in \left[1, p \frac{t-s'}{t}\right],$$

where $(\varphi_n)_{n \in \mathbb{N}}$ is any sequence in $\text{Lip}_b(G^{-1}(-\infty, 0))$ which converges in $W^{1,p}(G^{-1}(-\infty, 0), \nu)$ to φ . By Proposition 7.1 the definition does not depend on the choice of the sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $\text{Lip}_b(G^{-1}(-\infty, 0))$ approximating φ in $W^{1,p}(G^{-1}(-\infty, 0), \nu)$.

Observe that by [CL14, Proposition 4.8] $w\rho = w|_{G^{-1}(0)}\rho$. An obvious consequence of the definition is the following Corollary.

Corollary 7.3. Let $p \geq \frac{t}{t-s'}$. The operator

$$\text{Tr}_{G^{-1}(0)} : W^{1,p}(G^{-1}(-\infty, 0), \nu) \longrightarrow L^q(X, w\rho)$$

is continuous for every $q \in [1, p \frac{t-s'}{t}]$.

Proposition 7.4. *Let $a > 1$ and $b > 1$. If $\frac{ab}{a+b} \geq \frac{t}{t-s'}$, then for every $\varphi \in W^{1,a}(X, \nu)$ and $\psi \in W^{1,b}(X, \nu)$ we have*

$$\mathrm{Tr}_{G^{-1}(0)}(\varphi\psi) = \mathrm{Tr}_{G^{-1}(0)}(\varphi) \mathrm{Tr}_{G^{-1}(0)}(\psi) \quad \rho\text{-a.e.}$$

Proof. Let $(\varphi_n)_{n \in \mathbb{N}}, (\psi_n)_{n \in \mathbb{N}} \subseteq \mathrm{Lip}_b(G^{-1}(-\infty, 0))$ such that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \varphi_n &= \varphi && \text{in } W^{1,a}(G^{-1}(-\infty, 0), \nu); \\ \lim_{n \rightarrow +\infty} \psi_n &= \psi && \text{in } W^{1,b}(G^{-1}(-\infty, 0), \nu). \end{aligned}$$

By a standard argument we know $\varphi_n \psi_n \in \mathrm{Lip}_b(G^{-1}(-\infty, 0))$, for every $n \in \mathbb{N}$, and

$$\lim_{n \rightarrow +\infty} \varphi_n \psi_n = \varphi\psi \quad \text{in } W^{1, \frac{ab}{a+b}}(G^{-1}(-\infty, 0), \nu).$$

So we have

$$\mathrm{Tr}_{G^{-1}(0)}(\varphi\psi) = \lim_{n \rightarrow +\infty} \varphi_n|_{G^{-1}(0)} \psi_n|_{G^{-1}(0)} \quad \text{in } L^q(G^{-1}(0), w\rho) \text{ for every } q \in \left[1, \frac{ab}{a+b} \frac{t-s'}{t}\right].$$

Using Hölder inequality we get

$$\int_X |\varphi_n|_{G^{-1}(0)} \psi_n|_{G^{-1}(0)} - \mathrm{Tr}_{G^{-1}(0)}(\varphi) \mathrm{Tr}_{G^{-1}(0)}(\psi)| w d\rho \xrightarrow{n \rightarrow +\infty} 0.$$

So $\mathrm{Tr}_{G^{-1}(0)}(\varphi\psi) = \mathrm{Tr}_{G^{-1}(0)}(\varphi) \mathrm{Tr}_{G^{-1}(0)}(\psi)$ ρ -a.e. □

Proposition 7.5. *Let $p \geq \frac{t}{t-s'}$. For every $\varphi \in W^{1,p}(G^{-1}(-\infty, 0), \nu)$ we have*

$$\mathrm{Tr}_{G^{-1}(0)} \varphi(x) = \overline{\varphi}|_{G^{-1}(0)}(x) \quad \rho\text{-a.e.}$$

for every $(1, p)$ -precise version $\overline{\varphi}$ of φ .

Proof. Let $(\varphi_n)_{n \in \mathbb{N}}$ a sequence of bounded Lipschitz functions defined on $G^{-1}(-\infty, 0)$ which satisfies the condition of the Definition 7.2. By Proposition 6.4 we get

$$\begin{aligned} & \int_{G^{-1}(0)} \left| \overline{\varphi}|_{G^{-1}(0)} - \varphi_n \right| |\nabla_H G|_H w d\rho = \\ &= \int_{G^{-1}(-\infty, 0)} \mathrm{sign}(\overline{\varphi} - \varphi_n) \langle \nabla_H(\overline{\varphi} - \varphi_n), \nabla_H G \rangle_H - |\overline{\varphi} - \varphi_n| \mathrm{div}_\nu \nabla_H G d\nu. \end{aligned}$$

Letting $n \rightarrow +\infty$ we get $\mathrm{Tr}_{G^{-1}(0)} \varphi = \overline{\varphi}|_{G^{-1}(0)}$ ρ -a.e. □

We are now in position to prove Theorem 1.3.

Proof of Theorem 1.3. The statement follows by Proposition 6.3 and Corollary 7.5. The furthermore part follows by Proposition 7.4. □

Using the same argument we can extend Proposition 6.4 to functions in $W^{1,p}(G^{-1}(-\infty, 0), \nu)$.

Proposition 7.6. *Let $p \geq \frac{t}{t-s'}$ and $1 \leq q \leq p \frac{t-s'}{t}$. If $\varphi \in W^{1,p}(G^{-1}(-\infty, 0), \nu)$ then*

$$\begin{aligned} \int_{G^{-1}(-\infty, 0)} (q\varphi|\varphi|^{q-2} \langle \nabla_H \varphi, \nabla_H G \rangle_H - |\varphi|^q \operatorname{div}_\nu \nabla_H G) d\nu = \\ = \int_{G^{-1}(0)} \operatorname{Tr}_{G^{-1}(0)} |\varphi|^q \operatorname{Tr}_{G^{-1}(0)} |\nabla_H G|_H w d\rho, \end{aligned}$$

and

$$\int_{G^{-1}(-\infty, 0)} \left(q\varphi|\varphi|^{q-2} \left\langle \nabla_H \varphi, \frac{\nabla_H G}{|\nabla_H G|_H} \right\rangle + \operatorname{div}_\nu \frac{\nabla_H G}{|\nabla_H G|_H} |\varphi|^q \right) d\nu = \int_{G^{-1}(0)} (\operatorname{Tr}_{G^{-1}(0)} |\varphi|^q) w d\rho.$$

8. EXAMPLES

In this section we show how our results may be applied to some explicit examples. Recall that by $\partial_i f$ we denote the partial derivative of f along the direction $e_i \in H$ (see Equation (1.1)). We will be interested in two types of surfaces:

Unit sphere: Let $S(x) = \|x\|_X - 1$. We will prove that S satisfies Hypothesis 1.2 in everyone of our examples.

Hyperplanes: Let $f \in X^* \setminus \{0\}$ and $G_f(x) = f(x)$. Observe that $\partial_i G_f(x) = f(e_i)$ for every $i \in \mathbb{N}$ and $\partial_i \partial_j G_f(x) = 0$ for every $i, j \in \mathbb{N}$. Furthermore

$$G_f \in \bigcap_{p>1} W^{2,p}(X, \mu) \quad \text{and} \quad \frac{1}{|\nabla_H G_f|_H} \in \bigcap_{p>1} L^p(X, \mu).$$

Thus G_f satisfies Hypothesis 1.2, for every $f \in X^* \setminus \{0\}$.

8.1. A Gaussian-type weight in Hilbert spaces. Let X be a separable Hilbert space endowed with a nondegenerate centered Gaussian measure μ , with covariance Q . Fix an orthonormal basis $(v_n)_{n \in \mathbb{N}}$ of X of eigenvectors of Q , i.e. $Qv_k = \lambda_k v_k$, and the corresponding orthonormal basis of $H = Q^{\frac{1}{2}}(X)$ is $\{e_n := \sqrt{\lambda_n} v_n\}_{n \in \mathbb{N}}$.

Let $w_\lambda(x) = e^{\lambda(x,x)_X}$ for $\lambda \in \mathbb{R}$. Easy calculation gives

$$\partial_i w_\lambda(x) = 2\lambda(x, e_i)_X e^{\lambda(x,x)_X}, \quad \partial_i \log w_\lambda(x) = 2\lambda(x, e_i)_X, \quad \partial_j \partial_i \log w_\lambda(x) = 2\lambda(e_j, e_i)_X.$$

Let

$$\alpha := \sup \left\{ \eta > 0 \mid \int_X e^{\eta(x,x)_X} d\mu(x) < +\infty \right\}.$$

By Fernique's theorem (see [Bog98, Theorem 2.8.5]) the set $\{\eta > 0 \mid \int_X e^{\eta(x,x)_X} d\mu(x) < +\infty\}$ is not empty and α is strictly positive. Furthermore

$$\begin{aligned} \int_X e^{\eta(x,x)_X} d\mu(x) &= \int_X e^{\eta \sum_{i=1}^{+\infty} (x, v_i)_X^2} d\mu(x) = \lim_{n \rightarrow +\infty} \int_X e^{\eta \sum_{i=1}^n (x, v_i)_X^2} d\mu(x) = \\ &= \lim_{n \rightarrow +\infty} \frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^n \lambda_i}} \int_{\mathbb{R}^n} e^{\sum_{i=1}^n (\eta - \frac{1}{2\lambda_i}) \xi_i^2} d\xi = \lim_{n \rightarrow +\infty} \frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^n \lambda_i}} \prod_{i=1}^n \int_{\mathbb{R}} e^{(\eta - \frac{1}{2\lambda_i}) \xi_i^2} d\xi_i, \end{aligned}$$

and the last limit diverges if $\eta \geq 2\lambda_1$. Thus $0 < \alpha < 2\lambda_1$ and $w_\lambda \in W^{1,s}(X, \mu)$ for every $2\lambda < \alpha$ whenever $\lambda(s+1) \leq \alpha$. Furthermore $\log w_\lambda \in W^{2,t}(X, \mu)$ for every $t > 1$ and $2\lambda < \alpha$. In both cases it is possible to define $W^{1,p}(X, \nu)$ for every $p > 1$ (see Definition 4.3).

The above observation gives also that S satisfies Hypothesis 1.2. In this case $w_\lambda \rho = e^\lambda \rho$, so all the remarks about continuity of the trace operator in $L^p(S^{-1}(0), \rho)$ stated in [CL14, Section 5.3] still hold.

By Corollary 7.3, the trace operator $\text{Tr}_{G_f^{-1}(0)} : W^{1,p}(G_f^{-1}(-\infty, 0), \nu_\lambda) \rightarrow L^q(G_f^{-1}(0), w_\lambda \rho)$ for every $q \in [1, p)$ and $f \in X^* \setminus \{0\}$. Furthermore, using a similar argument as in Proposition 4.5, we get $\text{Tr}_{G_f^{-1}(0)} \varphi \in L^q(G_f^{-1}(0), \rho)$ for every $q \in [1, p)$ and every $\varphi \in W^{1,p}(G_f^{-1}(-\infty, 0), \nu)$.

8.2. A weight without continuous versions. Let $X = \ell_2$ the Banach space of square summable sequences and let $(v_k)_{k \in \mathbb{N}}$ its standard orthonormal basis, i.e. v_k is the sequence such that $v_k(i) = \delta_{ik}$ for every $i, k \in \mathbb{N}$. Let μ be a centered non-degenerate Gaussian measure on ℓ_2 with covariance operator $Q : \ell_2 \rightarrow \ell_2$ defined by

$$Q(x) = \left(\frac{x(i)}{2^i} \right).$$

Such a measure exists, e.g. by [Bog98, Theorem 2.3.1]. The eigenvectors of Q are the vectors v_k with respective eigenvalues 2^{-k} . We will denote by $\{e_n := v_n / \sqrt{2^n}\}$ the basis of the Cameron–Martin space associated with μ .

The weight we want to study is $w_q(x) = e^{\|x\|_q}$ for fixed $q > 1$. The first result we need is the fact that w_q (actually $\|\cdot\|_q$) is defined μ -a.e. and in order to show that we need a modification of Fernique’s theorem (see [Bog98, Theorem 2.8.5]). Let’s start with a definition:

Definition 8.1. Let γ be a Gaussian measure on a separable Banach space X and $p \in (0, 1]$. A function q measurable with respect to $\text{Borel}(X)$ (the Borel σ -algebra of X) is called *Borel(X)-measurable p -seminorm* if there exists a Borel(X)-measurable linear subspace $X_0 \subseteq X$ of γ -measure 1 such that q is a p -seminorm on X_0 , i.e. $q(x + y) \leq q(x) + q(y)$ for every $x, y \in X_0$ and $q(\lambda x) = |\lambda|^p q(x)$ for every $x \in X_0$ and $\lambda \in \mathbb{R}$.

Observe that our definition of Borel(X)-measurable 1-seminorm agrees with [Bog98, Definition 2.8.1]. Moreover our definition differs from [Bog98, Definition 3.9.2] since we also consider p -seminorm, with $0 < p < 1$.

Proposition 8.2. *Let μ be a centered Gaussian measure on a separable Banach space X , $p \in (0, 1]$ and let g be a Borel(X)-measurable p -seminorm. Then*

$$\int_X e^{\alpha g(x)^2} d\mu(x) < +\infty,$$

for some $\alpha > 0$.

Proof. Let $t > \tau > 0$. According to [Bog98, Proposition 2.2.10] we have

$$\begin{aligned} \mu(\{x \in X \mid g(x) \leq \tau\})\mu(\{y \in X \mid g(y) > t\}) &= \int_{\{(x,y) \in X \times X \mid g(x) \leq \tau \text{ and } g(y) > t\}} d\mu(x) d\mu(y) = \\ &= \int_{\{(u,v) \in X \times X \mid g\left(\frac{u-v}{\sqrt{2}}\right) \leq \tau, g\left(\frac{u+v}{\sqrt{2}}\right) > t\}} d\mu(u) d\mu(v) \leq \\ &\leq \int_{\{(u,v) \in X \times X \mid g(u) \geq \frac{t-\tau}{\sqrt{2^p}}, g(v) > \frac{t-\tau}{\sqrt{2^p}}\}} d\mu(u) d\mu(v). \end{aligned}$$

The last inequality follows by $q(u) \geq 2^{-p}(q(u+v) - q(u-v))$ for every $u, v \in X$, indeed

$$\left\{ (u, v) \in X^2 \left| g\left(\frac{u-v}{\sqrt{2}}\right) \leq \tau, g\left(\frac{u+v}{\sqrt{2}}\right) > t \right. \right\} \subseteq \left\{ (u, v) \in X^2 \left| g(u) \geq \frac{t-\tau}{\sqrt{2^p}}, g(v) > \frac{t-\tau}{\sqrt{2^p}} \right. \right\}.$$

Therefore we get

$$\mu(\{x \in X \mid g(x) \leq \tau\})\mu(\{y \in X \mid g(y) > t\}) \leq \left(\mu\left\{z \in X \left| g(z) > \frac{t-\tau}{\sqrt{2^p}} \right.\right\} \right)^2$$

Since $g < +\infty$ μ -a.e., there exists a positive number τ such that $c := \mu(\{x \in X \mid g(x) \leq \tau\}) > \frac{1}{2}$. If $c = 1$ the statement holds true, indeed $\int_X e^{\alpha g(x)^2} d\mu(x) \leq e^{\alpha \tau^2}$ for every $\alpha > 0$. Now assume $c < 1$. Let

$$t_n = \tau + \sqrt{2^p} t_{n-1} \quad \text{and} \quad t_0 = \tau.$$

It is easy to verify that $t_n = \tau(\sqrt{2^p}-1)^{-1}(2^{p\frac{n+1}{2}}-1)$. Letting $p_n := c^{-1}\mu(\{x \in X \mid g(x) > t_n\})$, then $p_n \leq p_{n-1}^2$. By induction we get

$$\mu(\{x \in X \mid g(x) > t_n\}) \leq c \left(\frac{1-c}{c} \right)^{2^n}.$$

Let $\alpha > 0$. We get

$$\begin{aligned} \int_X e^{\alpha g(x)^2} d\mu(x) &\leq \int_{\{x \in X \mid g(x) \leq \tau\}} e^{\alpha g(x)^2} d\mu(x) + \sum_{n=0}^{+\infty} e^{\alpha t_n^2} \mu(\{x \in X \mid t_n \leq g(x) < t_{n+1}\}) \leq \\ &\leq ce^{\alpha \tau^2} + \sum_{n=0}^{+\infty} e^{\alpha t_n^2} \mu(\{x \in X \mid g(x) \geq t_n\}) \leq ce^{\alpha \tau^2} + \sum_{n=0}^{+\infty} c \left(\frac{1-c}{c} \right)^{2^n} \exp\left(\alpha \tau^2 \frac{(2^{p\frac{n+1}{2}}-1)^2}{(\sqrt{2^p}-1)^2} \right) = \\ &= ce^{\alpha \tau^2} + c \sum_{n=0}^{+\infty} \exp\left(2^n \log\left(\frac{1-c}{c} \right) + \alpha \tau^2 \left(\frac{2^{p\frac{n+1}{2}}-1}{\sqrt{2^p}-1} \right)^2 \right) = \\ &= ce^{\alpha \tau^2} + c \sum_{n=0}^{[2p^{-1}(1-p)]} \exp\left(2^n \log\left(\frac{1-c}{c} \right) + \alpha \tau^2 \left(\frac{2^{p\frac{n+1}{2}}-1}{\sqrt{2^p}-1} \right)^2 \right) + \\ &+ c \sum_{n=[2p^{-1}(1-p)]+1}^{+\infty} \exp\left(2^n \log\left(\frac{1-c}{c} \right) + \alpha \tau^2 \left(\frac{2^{p\frac{n+1}{2}}-1}{\sqrt{2^p}-1} \right)^2 \right). \end{aligned}$$

Let $d = ce^{\alpha \tau^2} + c[2p^{-1}(1-p)] \exp\left(\log\left(\frac{1-c}{c} \right) + \alpha \tau^2 \left((\sqrt{2^p}-1)^{-1}(2^{p\frac{[2p^{-1}(1-p)]+1}}-1) \right)^2 \right)$, where $[\cdot]$ is the integer part function. Thus

$$\begin{aligned} \int_X e^{\alpha g(x)^2} d\mu(x) &\leq d + c \sum_{n=[2p^{-1}(1-p)]+1}^{+\infty} \exp\left(2^n \log\left(\frac{1-c}{c} \right) + \alpha \tau^2 \left(\frac{2^{p\frac{n+1}{2}}-1}{\sqrt{2^p}-1} \right)^2 \right) \leq \\ &\leq d + c \sum_{n=[2p^{-1}(1-p)]+1}^{+\infty} \exp\left(2^n \left(\log\left(\frac{1-c}{c} \right) + \alpha \tau^2 \left(\frac{2^{2p}-2^{\frac{3}{2}p}}{(\sqrt{2^p}-1)^2} \right) \right) \right). \end{aligned}$$

If we let $\alpha = \frac{(2^{\frac{p}{2}}-1)^2}{2\tau^2(2^{2p}-2^{\frac{3}{2}p})} \log \frac{c}{1-c}$, then the last series converges and the thesis follows. \square

Proposition 8.3. *Let $q > 0$ and $\ell_q = \{x \in \ell_2 \mid \sum_{i=1}^{+\infty} |(x, v_i)_{\ell_2}|^q < +\infty\}$ (observe that $\ell_q = \ell_2$ for every $q \geq 2$). Then $\mu(\ell_q) = 1$ and the function*

$$P_q(x) := \begin{cases} \sum_{i=1}^{+\infty} |(x, v_i)|^q & q \in (0, 1); \\ \|x\|_q & q \geq 1, \end{cases}$$

belongs to $L^p(\ell_2, \mu)$, for every $p \geq 1$.

Proof. By [Bog98, Exercise A.3.34], for every $q > 0$ the linear space ℓ_q is measurable with respect to the Borel σ -algebra in ℓ_2 . We have

$$\int_{\ell_2} \sum_{i=1}^n |(x, v_i)|^q d\mu(x) = \sum_{i=1}^n \int_{\ell_2} |(x, v_i)|^q d\mu(x).$$

Using the change of variable formula (see [Bog98, Equation (A.3.1)]) we get

$$\int_{\ell_2} \sum_{i=1}^n |(x, v_i)|^q d\mu(x) = \sum_{i=1}^n \frac{2\sqrt{2^{i-1}}}{\sqrt{\pi}} \int_0^{+\infty} \eta^q e^{-2^{i-1}\eta^2} d\eta.$$

Let $c_q = 2\sqrt{\pi^{-1}} \int_0^{+\infty} t^q e^{-t^2} dt$. Then

$$\int_{\ell_2} \sum_{i=1}^n |(x, v_i)|^q d\mu(x) = c_q \sum_{i=1}^n \left(\frac{1}{2^{\frac{q}{2}}}\right)^i = c_q \frac{2^{\frac{q}{2}(n+1)} - 1}{2^{\frac{q}{2}n}(2^{\frac{q}{2}} - 1)} \leq c_q \frac{2^{\frac{q}{2}}}{2^{\frac{q}{2}} - 1}.$$

Since $\sum_{i=1}^n |(x, v_i)| v_i \rightarrow x$ μ -a.e., then by the Lebesgue dominated convergence theorem we get

$$\int_{\ell_2} \sum_{i=1}^{+\infty} |(x, v_i)|^q d\mu(x) < +\infty.$$

By [Bog98, Theorem 2.5.5] we have $\mu(\ell_q) = 1$. Observe that P_q is a Borel(X)-measurable q -norm, for every $q > 0$, then by Theorem 8.2 it belongs to $L^p(\ell_2, \mu)$ for every $p \geq 1$. \square

Proposition 8.3 implies that $w_q = e^{\|x\|_q}$ is defined μ -a.e. on ℓ_2 for every $q > 1$. Now we need to prove that w_q satisfies Hypothesis 1.1. We start with another modification of Fernique's theorem (see [Bog98, Theorem 2.8.5]), which implies that for every $q > 1$, $w_q \in L^s(\ell_2, \mu)$ for every $s \geq 1$.

Proposition 8.4. *Let μ be a centered Gaussian measure on a locally convex space X and let g be a measurable 1-seminorm. Then $\int_X e^{\alpha g(x)} d\mu(x) < +\infty$, for every $\alpha > 0$.*

Proof. Let $t > \tau > 0$. According to [Bog98, Proposition 2.2.10] we have

$$\begin{aligned} \mu(\{x \in X \mid g(x) \leq \tau\}) \mu(\{y \in X \mid g(y) > t\}) &= \int_{\{(x,y) \in X \times X \mid g(x) \leq \tau, g(y) > t\}} d\mu(x) d\mu(y) = \\ &= \int_{\{(u,v) \in X \times X \mid g\left(\frac{u-v}{\sqrt{2}}\right) \leq \tau, g\left(\frac{u+v}{\sqrt{2}}\right) > t\}} d\mu(u) d\mu(v) \leq \\ &\leq \int_{\{(u,v) \in X \times X \mid g(u) \geq \frac{t-\tau}{\sqrt{2}}, g(v) > \frac{t-\tau}{\sqrt{2}}\}} d\mu(u) d\mu(v). \end{aligned}$$

The last inequality follows by $q(u) \geq 2^{-1}(q(u+v) - q(u-v))$ for every $u, v \in X$ and the fact that

$$\left\{ (u, v) \in X^2 \left| g\left(\frac{u-v}{\sqrt{2}}\right) \leq \tau, g\left(\frac{u+v}{\sqrt{2}}\right) > t \right. \right\} \subseteq \left\{ (u, v) \in X^2 \left| g(u) \geq \frac{t-\tau}{\sqrt{2}}, g(v) > \frac{t-\tau}{\sqrt{2}} \right. \right\}.$$

Therefore we get

$$(8.1) \quad \mu(\{x \in X \mid g(x) \leq \tau\})\mu(\{y \in X \mid g(y) > t\}) \leq \left(\mu\left\{z \in X \left| g(z) > \frac{t-\tau}{\sqrt{2}} \right.\right\} \right)^2$$

Since $g < +\infty$ μ -a.e., there exists a positive number τ such that $c := \mu(\{x \in X \mid g(x) \leq \tau\}) > \frac{1}{2}$. If $c = 1$ the statement holds true, indeed $\int_X e^{\alpha g(x)} d\mu(x) \leq e^{\alpha\tau}$ for every $\alpha > 0$. Now assume $c < 1$. Let

$$t_n = \tau + \sqrt{2}t_{n-1} \quad \text{and} \quad t_0 = \tau.$$

It is easy to verify $t_n = \tau(1 + \sqrt{2})(2^{\frac{n+1}{2}} - 1)$. Letting $p_n := c^{-1}\mu(\{x \in X \mid g(x) > t_n\})$, it is easy to prove that $p_n \leq p_{n-1}^2$. By induction we get

$$\mu(\{x \in X \mid g(x) > t_n\}) \leq c \left(\frac{1-c}{c} \right)^{2^n}.$$

For every $\alpha > 0$ we get

$$\begin{aligned} \int_X e^{\alpha g(x)} d\mu(x) &\leq \int_{\{x \in X \mid g(x) \leq \tau\}} e^{\alpha g(x)} d\mu(x) + \sum_{n=0}^{+\infty} e^{\alpha t_{n+1}} \mu(\{x \in X \mid t_n \leq g(x) < t_{n+1}\}) \leq \\ &\leq \int_{\{x \in X \mid g(x) \leq \tau\}} e^{\alpha g(x)} d\mu(x) + \sum_{n=0}^{+\infty} e^{\alpha t_{n+1}} \mu(\{x \in X \mid g(x) \geq t_n\}) \leq \\ &\leq \int_{\{x \in X \mid g(x) \leq \tau\}} e^{\alpha g(x)} d\mu(x) + \sum_{n=0}^{+\infty} c \left(\frac{1-c}{c} \right)^{2^n} e^{\alpha \tau (1+\sqrt{2}) (2^{\frac{n+2}{2}} - 1)} = \\ &= \int_{\{x \in X \mid g(x) \leq \tau\}} e^{\alpha g(x)} d\mu(x) + c \sum_{n=0}^{+\infty} e^{2^n \log\left(\frac{1-c}{c}\right) + \alpha \tau (1+\sqrt{2}) (2^{\frac{n+2}{2}} - 1)}. \end{aligned}$$

The last series converges. □

Proposition 8.5. *For every $q > 1$ consider the function $U_q : \ell_2 \rightarrow \mathbb{R}$ defined by*

$$U(x) = \begin{cases} \|x\|_q & x \in \ell_q; \\ 0 & x \notin \ell_q. \end{cases}$$

Then $U \in W^{1,p}(\ell_2, \mu)$ for every $p \geq 1$ and

$$\partial_i U(x) = 2^{-\frac{i}{2}} \text{sign}(x, v_i) (x, v_i)^{q-1} \|x\|_q^{1-q} \quad \mu\text{-a.e.}$$

Proof. For every $n \in \mathbb{N}$ consider the function $\varphi_n : \mathbb{R}^n \rightarrow \mathbb{R}$ defined as

$$\varphi_n(\eta_1, \dots, \eta_n) := \left(\sum_{i=1}^n \left(\eta_i^2 + \frac{1}{2^n} \right)^{\frac{q}{2}} \right)^{\frac{1}{q}}.$$

Let $U_n(x) := \varphi_n((x, v_1), \dots, (x, v_n))$ and observe that $U_n \rightarrow U$ pointwise μ -a.e. and $U_n \in \mathcal{FC}_b^\infty(\ell_2)$ for every $n \in \mathbb{N}$. Indeed for every $n \geq 2$

$$\begin{aligned} \left(\sum_{i=1}^n |(x, v_i)|^q \right)^{\frac{1}{q}} &\leq |U_n(x)| \leq \left(\max \{1, 2^{q-1}\} \sum_{i=1}^n \left(|(x, v_i)|^q + \frac{1}{2^{\frac{nq}{2}}} \right) \right)^{\frac{1}{q}} \leq \\ &\leq \max \left\{ 1, 2^{\frac{q-1}{q}} \right\} \left(\frac{n}{2^{\frac{nq}{2}}} + \sum_{i=1}^n |(x, v_i)|^q \right)^{\frac{1}{q}} \leq \max \left\{ 1, 2^{\frac{q-1}{q}} \right\} \left(\frac{n}{2^{\frac{n}{2}}} + \left(\sum_{i=1}^n |(x, v_i)|^q \right)^{\frac{1}{q}} \right) \\ &\leq \max \left\{ 1, 2^{\frac{q-1}{q}} \right\} (1 + \|x\|_q). \end{aligned}$$

By Lebesgue's dominated convergence theorem and Proposition 8.3 we get $U_n \rightarrow U$ in $L^p(\ell_2, \mu)$ for every $p \geq 1$. Observe that

$$\partial_i U_n(x) = \frac{(x, v_i)}{2^{\frac{i}{2}}} \left((x, v_i)^2 + \frac{1}{2^n} \right)^{\frac{q}{2}-1} \left(\sum_{k=1}^n \left((x, v_k)^2 + \frac{1}{2^n} \right)^{\frac{q}{2}} \right)^{\frac{1}{q}-1},$$

and $\partial_i U_n(x) \rightarrow 2^{-\frac{i}{2}} \text{sign}(x, v_i) (x, v_i)^{q-1} \|x\|_q^{1-q}$ pointwise μ -a.e. For $1 < q \leq 2$

$$\begin{aligned} |\nabla_H U_n(x)|_H &= \left(\sum_{i=1}^{+\infty} \frac{(x, v_i)^2}{2^i} \left((x, v_i)^2 + \frac{1}{2^n} \right)^{q-2} \left(\sum_{k=1}^n \left((x, v_k)^2 + \frac{1}{2^n} \right)^{\frac{q}{2}} \right)^{\frac{2}{q}-2} \right)^{\frac{1}{2}} \leq \\ &\leq 2^{\frac{q-1}{2}} \left(\sum_{i=1}^{+\infty} \frac{(x, v_i)^2}{2^i} \left((x, v_i)^2 + \frac{1}{2^n} \right)^{q-2} \right)^{\frac{1}{2}} \leq 2^{\frac{q-1}{2}} \left(\sum_{i=1}^{+\infty} \frac{(x, v_i)^{2q-2}}{2^i} \right)^{\frac{1}{2}} \leq 2^{\frac{q-1}{2}} \left(\sum_{i=1}^{+\infty} (x, v_i)^{q-1} \right). \end{aligned}$$

By Proposition 8.3 the last function is integrable for every $p \geq 1$. If $p > 2$ then

$$\begin{aligned} |\nabla_H U_n(x)|_H &= \left(\sum_{i=1}^{+\infty} \frac{(x, v_i)^2}{2^i} \left((x, v_i)^2 + \frac{1}{2^n} \right)^{q-2} \left(\sum_{k=1}^n \left((x, v_k)^2 + \frac{1}{2^n} \right)^{\frac{q}{2}} \right)^{\frac{2}{q}-2} \right)^{\frac{1}{2}} \leq \\ &\leq 2^{\frac{q-1}{2}} \left(\sum_{i=1}^{+\infty} \frac{(x, v_i)^2}{2^i} \left((x, v_i)^2 + \frac{1}{2^n} \right)^{q-2} \right)^{\frac{1}{2}} \leq \\ &\leq 2^{\frac{q-1}{2}} \max \left\{ 1, 2^{\frac{q-3}{2}} \right\} \left(\sum_{i=1}^{+\infty} \left((x, v_i)^{2q-2} + \frac{(x, v_i)^4}{2^{n(q-2)}} \right) \right)^{\frac{1}{2}} \leq \\ &\leq 2^{\frac{q-1}{2}} \max \left\{ 1, 2^{\frac{q-3}{2}} \right\} \left(\sum_{i=1}^{+\infty} (x, v_i)^{q-1} + \sum_{i=1}^{+\infty} (x, v_i)^2 \right). \end{aligned}$$

By Proposition 8.3 the last term belongs to $L^p(\ell_2, \mu)$ for every $p \geq 1$. By the Lebesgue dominated convergence theorem we get $U \in W^{1,p}(\ell_2, \mu)$ for every $p \geq 1$ and $\partial_i U(x) = 2^{-\frac{i}{2}} \text{sign}(x, v_i) (x, v_i)^{q-1} \|x\|_q^{1-q}$ μ -a.e. for every $i \in \mathbb{N}$. \square

By Proposition 4.6 (applied with the functions $\theta_n(\eta) = n \arctan(n^{-1}e^\eta)$, for every $\eta \in \mathbb{R}$ and $n \in \mathbb{N}$, and then using the Lebesgue dominated convergence theorem) we get

$$\begin{aligned}\partial_i w_q(x) &= 2^{-\frac{i}{2}} \operatorname{sign}(x, v_i)(x, v_i)^{q-1} \|x\|_q^{1-q} e^{\|x\|_q}; \\ \partial_i \log w_q(x) &= 2^{-\frac{i}{2}} \operatorname{sign}(x, v_i)(x, v_i)^{q-1} \|x\|_q^{1-q}.\end{aligned}$$

Proposition 8.4 and Proposition 8.5 yield that the function w_q satisfies Hypothesis 1.1 for every $s, t > 1$. This implies that it is possible to define $W^{1,p}(\ell_2, \nu_q)$ for every $p > 1$ (see Definition 4.3). Observe that $\|\cdot\|_q$ is not μ -a.e. continuous on ℓ_2 if $q \in (1, 2)$.

Using the same argument already used in Example 8.1 it is possible to prove that S satisfies Hypothesis 1.2. The trace operator $\operatorname{Tr}_{S^{-1}(0)}$ maps $W^{1,p}(S^{-1}(-\infty, 0), \nu_q)$ into $L^r(S^{-1}(0), w_q \rho)$ for every $r \in [1, p)$. We do not know whether its range is contained in $L^p(S^{-1}(0), w_q \rho)$. The same considerations are true for the trace operator $\operatorname{Tr}_{G_f^{-1}(0)}$ for every $f \in \ell_2 \setminus \{0\}$.

8.3. An example in the space of continuous functions. Recall that a function $f : X \rightarrow \mathbb{R}$ from a Banach space X to \mathbb{R} is Gâteaux differentiable in $x \in X$ if for every $y \in X$ the limit

$$\lim_{t \rightarrow 0} \frac{f(x + ty) - f(x)}{t}$$

exists and defines a linear (in y) map $((Df)x)(\cdot)$ which is continuous from X to \mathbb{R} .

We will use the following result of Aronszajn (see [Aro76, Theorem 1 of Chapter 2] and [Phe78, Theorem 6]).

Theorem 8.6. *Suppose that X is a separable real Banach space. If $f : X \rightarrow \mathbb{R}$ is a continuous convex function, then f is Gâteaux differentiable outside of a Gaussian null set, i.e. a Borel set $A \subseteq X$ such that $\mu(A) = 0$ for every nondegenerate Gaussian measure μ on X .*

Consider the classical Wiener measure P^W on $\mathcal{C}[0, 1]$ (see [Bog98, Example 2.3.11 and Remark 2.3.13] for its construction). Recall that the Cameron–Martin space is the space of the continuous functions f on $[0, 1]$ such that f is absolutely continuous, $f' \in L^2[0, 1]$ and $f(0) = 0$. In addition $\|f\|_H = \|f'\|_{L^2[0, 1]}$ (see [Bog98, Lemma 2.3.14]). An orthonormal basis of H is given by the functions

$$f_n(\xi) = \sqrt{2} \sin \frac{\xi}{\sqrt{\lambda_n}} \quad \text{where } \lambda_n = \frac{4}{\pi^2(2n-1)^2} \text{ for every } n \in \mathbb{N}.$$

Consider the weight $w(f) = e^{\|f\|_\infty}$. According to [Bog98, Theorem 5.11.2] w is differentiable along H . In particular letting

$$M = \{f \in \mathcal{C}[0, 1] \mid \text{there exists a unique } \xi \in [0, 1] \text{ such that } \|f\|_\infty = |f(\xi)|\},$$

by Theorem 8.6 and [DGZ93, Example 1.6.b] we get that $P^W(M) = 1$. Furthermore by [Ban87]

$$((D\|\cdot\|_\infty)f)(g) = \operatorname{sign}(f(\xi_f))g(\xi_f),$$

for every $f \in M$ and $g, g_1, g_2 \in \mathcal{C}[0, 1]$, where $\xi_f \in [0, 1]$ is the only point of maximum of the function $|f(\cdot)|$. By [Bog98, Definition 5.2.3 and Proposition 5.4.6(iii)] and by Proposition 4.6

(applied with the functions $\theta_n(\eta) = n \arctan(n^{-1}e^\eta)$, for every $\eta \in \mathbb{R}$ and $n \in \mathbb{N}$, and then using the Lebesgue dominated convergence theorem) it can be seen that

$$\begin{aligned} (\partial_i w(f))(\xi) &= e^{\|f\|_\infty} \text{sign}(f(\xi_f)) f_i(\xi_f); \\ (\partial_i \log w(f))(\xi) &= \text{sign}(f(\xi_f)) f_i(\xi_f). \end{aligned}$$

We also have

$$(8.2) \quad \|w\|_{L^s(\mathcal{C}[0,1], P^W)} = \left(\int_{\mathcal{C}[0,1]} e^{s\|f\|_\infty} dP^W(f) \right)^{\frac{1}{s}};$$

$$(8.3) \quad \|\nabla_H w\|_{L^s(\mathcal{C}[0,1], P^W; H)} = \left(\int_{\mathcal{C}[0,1]} e^{s\|f\|_\infty} \left(\sum_{n=1}^{+\infty} f_n^2(\xi_f) \right)^s dP^W(f) \right)^{\frac{1}{s}};$$

$$(8.4) \quad \|\log w\|_{L^t(\mathcal{C}[0,1], P^W)} = \left(\int_{\mathcal{C}[0,1]} \|f\|_\infty^t dP^W(f) \right)^{\frac{1}{t}};$$

$$(8.5) \quad \|\nabla_H \log w\|_{L^t(\mathcal{C}[0,1], P^W; H)} = \left(\int_{\mathcal{C}[0,1]} \left(\sum_{n=1}^{+\infty} f_n^2(\xi_f) \right)^t dP^W(f) \right)^{\frac{1}{t}};$$

By Theorem 8.4, Equations (8.2) and (8.4) are finite for every $s, t > 1$. By [Bog98, Theorem 5.11.2], Equations (8.3) and (8.5) are finite for every $s > 1$. All these results give that the weight w satisfies Hypothesis 1.1 for every $s, t > 1$. This implies that it is possible to define $W^{1,p}(\mathcal{C}[0,1], \nu)$ for every $p > 1$ (see Definition 4.3).

If we let $S_1(f) = \|f\|_2 - 1$, then

$$S_1 \in \bigcap_{p>1} W^{2,p}(\mathcal{C}[0,1], P^W).$$

Furthermore $\nabla_H S_1(f) = \sum_{i=1}^{+\infty} \frac{\int_0^1 f(\xi) f_i(\xi) d\xi}{\|f\|_2} f_i$ and

$$(8.6) \quad \int_{\mathcal{C}[0,1]} \frac{1}{|\nabla_H S_1(f)|_H^p} dP^W(f) = \int_{\mathcal{C}[0,1]} \frac{\|f\|_2^p}{(\sum_{j=1}^{+\infty} (\int_0^1 f(\xi) f_n(\xi) d\xi)^2)^p} dP^W(f).$$

Using the Hölder inequality with some $\alpha > 1$ we get

$$\int_{\mathcal{C}[0,1]} \frac{1}{|\nabla_H S_1(f)|_H^p} dP^W(f) \leq \left(\int_{\mathcal{C}[0,1]} \|f\|_2^{p\alpha'} dP^W(f) \right)^{\frac{1}{\alpha'}} \left(\int_{\mathcal{C}[0,1]} \frac{dP^W(f)}{(\sum_{j=1}^{+\infty} (\int_0^1 f(\xi) f_n(\xi) d\xi)^2)^{p\alpha}} \right)^{\frac{1}{\alpha}}.$$

Finally $\int_{\mathcal{C}[0,1]} \|f\|_2^{p\alpha'} dP^W(f)$ is finite by Fernique's theorem (see [Bog98, Theorem 2.8.5]), and recalling that L^2 -continuous linear functionals are also $\mathcal{C}[0,1]$ -continuous we can use the change of variable formula (see [Bog98, Equation (A.3.1)]) and obtain

$$\int_{\mathcal{C}[0,1]} \frac{dP^W(f)}{(\sum_{j=1}^{+\infty} (\int_0^1 f(\xi) f_n(\xi) d\xi)^2)^{p\alpha}} \leq \int_{\mathcal{C}[0,1]} \frac{dP^W(f)}{(\sum_{j=1}^N (\int_0^1 f(\xi) f_n(\xi) d\xi)^2)^{p\alpha}} = \int_{\mathbb{R}^N} \frac{d\mu_N(\xi)}{(\sum_{j=1}^N \xi_j^2)^{p\alpha}},$$

where μ_N denote the N -dimensional Gaussian measure. For $N \in \mathbb{N}$ big enough the last integral is finite and we get that (8.6) is finite for every $p > 1$. Thus S_1 satisfies Hypothesis

1.2. By Corollary 7.3, the trace operator $\text{Tr}_{S^{-1}(0)}$ maps the space $W^{1,p}(S_1^{-1}(-\infty, 0), \nu)$ into $L^q(S_1^{-1}(0), w\rho)$ continuously, for every $q \in [1, p)$.

Fix a finite Borel measure λ on $[0, 1]$ and recall that $G_\lambda(f) = \int_0^1 f(x)d\lambda(x)$. Using a similar argument as in Proposition 4.5, we get $\text{Tr}_{G_\lambda^{-1}(0)} \varphi \in L^q(G_\lambda^{-1}(0), \rho)$ for every $q \in [1, p)$ and every $\varphi \in W^{1,p}(G_\lambda^{-1}(-\infty, 0), \nu)$.

REFERENCES

- [AM88] H. Airault and P. Malliavin. Intégration géométrique sur l'espace de wiener. *Bull. Sci. Math.*, 112:3–52, 1988.
- [AMP10] Luigi Ambrosio, Michele Miranda, Jr., and Diego Pallara. Sets with finite perimeter in Wiener spaces, perimeter measure and boundary rectifiability. *Discrete Contin. Dyn. Syst.*, 28(2):591–606, 2010.
- [Aro76] N. Aronszajn. Differentiability of Lipschitzian mappings between Banach spaces. *Studia Math.*, 57(2):147–190, 1976.
- [Ban87] S. Banach. *Theory of linear operations*, volume 38 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1987. Translated from the French by F. Jellett, With comments by A. Pełczyński and Cz. Bessaga.
- [Bog98] Vladimir I. Bogachev. *Gaussian measures*, volume 62 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.
- [CL14] Pietro Celada and Alessandra Lunardi. Traces of Sobolev functions on regular surfaces in infinite dimensions. *J. Funct. Anal.*, 266(4):1948–1987, 2014.
- [DGZ93] Robert Deville, Gilles Godefroy, and Václav Zizler. *Smoothness and renormings in Banach spaces*, volume 64 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1993.
- [DPL14] Giuseppe Da Prato and Alessandra Lunardi. Sobolev regularity for a class of second order elliptic PDE's in infinite dimension. *Ann. Probab.*, 42(5):2113–2160, 2014.
- [DU77] J. Diestel and J. J. Uhl, Jr. *Vector measures*. American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.
- [Dug78] James Dugundji. *Topology*. Allyn and Bacon, Inc., Boston, Mass.-London-Sydney, 1978. Reprinting of the 1966 original, Allyn and Bacon Series in Advanced Mathematics.
- [FdLP91] D. Feyel and A. de La Pradelle. Capacités gaussiennes. *Ann. Inst. Fourier (Grenoble)*, 41(1):49–76, 1991.
- [Fey01] D. Feyel. Hausdorff-Gauss measures. In *Stochastic analysis and related topics, VII (Kusadasi, 1998)*, volume 48 of *Progr. Probab.*, pages 59–76. Birkhäuser Boston, Boston, MA, 2001.
- [Hin03] Masanori Hino. On Dirichlet spaces over convex sets in infinite dimensions. In *Finite and infinite dimensional analysis in honor of Leonard Gross (New Orleans, LA, 2001)*, volume 317 of *Contemp. Math.*, pages 143–156. Amer. Math. Soc., Providence, RI, 2003.

- [Hin11] Masanori Hino. Dirichlet spaces on H -convex sets in Wiener space. *Bull. Sci. Math.*, 135(6-7):667–683, 2011.
- [Man90] Mark Mandelkern. On the uniform continuity of Tietze extensions. *Arch. Math. (Basel)*, 55(4):387–388, 1990.
- [McS34] E. J. McShane. Extension of range of functions. *Bull. Amer. Math. Soc.*, 40(12):837–842, 1934.
- [Mic03] Radu Miculescu. Approximations by Lipschitz functions generated by extensions. *Real Anal. Exchange*, 28(1):33–40, 2002/03.
- [Phe78] R. R. Phelps. Gaussian null sets and differentiability of Lipschitz map on Banach spaces. *Pacific J. Math.*, 77(2):523–531, 1978.
- [Rud87] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

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